B. S. Belikov

General methods for solving physics problems

Mir Publishers Moscow

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Б. С. Беликов

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TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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PREFACE

The revolution in science and engineering makes it necessary to seek new ways of more effectively training specialists for the national economy. What must be done so that the theoretical knowledge that a student acquires in college does not remain passive but is used with maximal efficiency in practical work? Until recently this was achieved with relative success in the traditional way, but now with the information explosion this is becoming ever more difficult. The body of information has become so large that it cannot be comprehended in the limited period of education unless organized on an entirely new basis. This basis could be the extended and systematic use of generalized methods, general methodological principles, and very general concepts.

This book attempts to do so in a segment of student instruction of vital importance, the solution of physics problems. The approach is based on the application of the most general concepts of physics to the solution of any problem. I consider the theoretical aspects underlying the general approach to problem solution and methods for solving standard, nonstandard, nonspecific, and general problems.

Such an approach is not the only one, of course. Many problems can be solved much faster by applying nonstandard methods or by intuition. Nevertheless, a nonstandard approach and the use of intuition are possible only when the student possesses a methodological base for problem solving and has mastered the general approach. This book may, therefore, prove helpful for students of diversified specialities.

Some of the problems discussed here have been taken from A.G. Chertov and A.A. Vorob'ev, *A Problem Book in Physics* (Moscow, 1981, in Russian), and I.E. Irodov,

Problems in General Physics (Moscow, 1979, English translation: Mir Publishers, Moscow, 1981, 2nd ed. 1983). The ideas incorporated in others belong to me.

I take this opportunity to thank Assistant Professors V.B. Zernov and A.F. Menyaev and also the staff of the Physics Faculty of the Moscow Institute of Electronic Machinery Industry for their constructive comments to improve this book.

B.S. Belikov

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INTRODUCTION

It often happens that a student has a good knowledge of the theoretical aspects of the physics course but does not know how to solve physics problems. Some students admit that they encounter no difficulties in studying theory and can memorize and understand formulas, definitions, etc., but are not able to cope with problems. They do not even know how to begin, and at times, while solving a problem, they get bogged down in theory and after writing down numerous formulas, laws, and equations do not know whether they have solved the problem or are close to or far from the solution. Often, after solving a problem correctly in general form, they make mistakes in calculations, and an incorrect numerical result means an incorrect solution, after all, and cannot be considered valid.

Granted: it is not easy to learn to solve physics problems. One can know theory well and yet not be able to solve the simplest problem. This is no accident. Knowing theory is necessary but not sufficient for problem solving. Besides specific knowledge one must possess *generalized knowledge*. And this is usually acquired through experience, in the process of solving physics problems, usually toward the end of the physics course. But sometimes generalized knowledge is not acquired at all.

The body of generalized knowledge necessary for problem solving is discussed mainly in the first two chapters of this book. The reader will learn what a *physics problem* is, from what angle to approach it, what signifies that a solution has been found, and much more.

The base of generalized knowledge is the *fundamental concepts* of physics, which are methodological in character. These are discussed in the first chapter.

There are relatively few fundamental methodological concepts of physics. This book uses eight: a *physical sys-*

tem, a *physical quantity*, a *physical law*. the *state of a physical system*, *interaction*, a *physical phenomenon*, *idealized objects* and *processes* and a *physical model*. Being interrelated, these concepts form a *system*. Of special importance is the relation of a physical phenomenon to all other fundamental concepts.

Using a system of fundamental concepts makes it possible to formulate the most important definition of a theoretical physical problem as a *physical phenomenon in which some of the relationships and quantities are unknown*.

> To solve a physics problem means to establish the unknown relationships and determine the sought physical quantities.

This definition is extremely important methodologically. If a physics problem reflects a physical phenomenon (or collection of phenomena), it is necessary not only to have specific knowledge about this phenomenon but also to know how to analyze any physical phenomenon by applying generalized knowledge. This analysis begins with the choice and analysis of a physical system and ends with the construction of a closed system of equations by the application of the appropriate physical laws. In view of this the solution process breaks down into three stages: the *physical* (which ends with the construction of the closed system of equations), the *mathematical* (designed to obtain a solution in general and numerical forms), and *analysis* of this solution.

This, naturally, requires building a system of methods (discussed in Chapter 2) for solving physics problems as a system of general guidelines for each of the three stages.

It is believed that no one method for solving all physics problems exists. This may be true. I believe, however, there does exist a general approach (system of methods) to the solution of any physics problem.

In Chapters 3 to 13 the general approach is applied to the solution of problems in practically all the classical divisions of the general physics course.

Part 1

THE THEORETICAL BASES OF THE GENERAL APPROACH TO SOLVING ANY PHYSICS PROBLEM

Chapter 1

THE SYSTEM OF FUNDAMENTAL CONCEPTS OF PHYSICS

1. Some General Concepts of Physics

The general approach to solving any physics problem is based on several fundamental physical concepts. These are well known, and we mention them in this section only to note some of their specific features. One central concept is that of a physical system.

> A physical system constitutes a collection of physical objects or even a single physical object.

The solution of any physics problem is related to the study of a particular system. In what follows we will see that the choice and investigation of a physical system constitute the beginning of an analysis of the physics of a problem.

The physical objects in a system possess certain physical properties and can participate in various physical processes. The common way to characterize the properties of physical objects and processes is to introduce various *physical quantities*.

Another important concept is that of the *state of a physical system*. The general concept of the state of any physical system is fairly complicated. If a physical system consists of a single particle, its mechanical state is determined by six quantities: three position coordinates (x, y, z) and three components of the particle's momentum (p_x, p_y, p_z).

The objects in a physical system are interrelated both with each other and with external objects. This interrelationship manifests itself in the interaction of physical objects.

Interaction constitutes one of the most important properties of any physical objects. It is caused by the inner nature of the objects. Four basic types of interaction between elementary particles are known in physics: *strong*, *electromagnetic*, *weak*, and *gravitational*. Neither strong nor weak interactions will be touched on in this book.

Interaction may change either the position of a physical system or its state.

The change in position or state of a physical system will be called a physical phenomenon.

Within the framework of the general approach to solving physics problems this concept is the most important. Nature knows of a great variety of physical phenomena: the movements of planets and stars, thunder and lightning, rain and wind, water evaporation and dew fall, the vibration of atoms and the movement of molecules, the emission and absorption of light, and so on. It is not so simple to understand even the qualitative aspect of one or another physical phenomenon. ·

It has proved expedient to start analyzing a physical phenomenon by selecting and investigating the appropriate physical system, since the phenomenon proper always occurs within a system.

In the process of analyzing the physical objects of the system it is advisable to establish to which ideal objects they correspond, what properties they possess, with what objects and in what manner they may interact, and what the results and consequences of such interaction may be.

A physical phenomenon is characterized by a change in certain physical quantities. These are interrelated.

As is known, a necessary and stable interrelation or dependence between physical quantities is expressed by a physical law.

Every physical law has many facets or angles (necessity, objectivity, physical meaning, and so on). In what follows we will have special need for two specific aspects of a physical law: the *conditions* (or limits) *of its applicability* and the *method of application*.

Every physical law is relative in the sense that it is valid only in certain conditions.

> The set of these limitations will be called the conditions (or limits) of applicability of the given law.

If any one of these conditions is violated, the law cannot be employed, it becomes invalid. For example, Newton's second law written in the form $\mathbf{F} = m\mathbf{a}$ is valid if the following conditions are maintained: the motion of an object is considered in relation to an inertial reference frame, the object is a particle, the particle mass is constant, the velocity of the object (or particle) is much smaller than the speed of light, etc. If any one of these conditions is violated, Newton's second law in the above form cannot be employed.

When solving physics problems, it is not enough to know the right law (i.e. its physical meaning, conditions of applicability, etc.): one must also know how to apply the law in specific conditions.

> Every physical law has its method (algorithm) of application.

Say, to write Newton's second law in the form $\mathbf{F} = m\mathbf{a}$ correctly, one must perform the following sequence of operations. First, check to see whether the applicability conditions for the law are met (if any one is violated, the law cannot be employed). Second, select an inertial reference frame (the given law is valid only in such reference frames). Third, find all the forces acting on the object under investigation (the law incorporates the physical quantity $\mathbf{F}$, which constitutes the vector sum of all the forces acting on the object of mass m). Fourth, determine the projection of each force on the coordinate axes (Newton's second law is a vector law). Fifth, sum algebraically the projections of the forces on each coordinate axis ($\sum F_x$, $\sum F_y$, $\sum F_z$). And finally, sixth,

write Newton's second law in the form of three equations

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z,$$

where a_x, a_y, and a_z are the projections of acceleration **a** on the x, y, and z axes. The methods used in applying other physical laws will be discussed in the appropriate chapters.

2. Idealization of a Physics Problem

Suppose that a formulated problem contains the necessary data (whose completeness is ensured) and we must determine certain unknown physical quantities. But this is not the main thing in a formulated problem. The most important aspect is that the problem is already idealized. The author of the problem has introduced many additional conditions simplifying it. By introducing these conditions and limitations, he has artificially severed the links of the given physical phenomenon with other phenomena. It is also assumed that the influence of other (additional) phenomena is small and can be ignored. Thus,

> a formulated physics problem is a problem concerned with a pure, idealized phenomenon.

Very often in science the object of investigation is not a real object but its ideal image. The explanation lies in the fact that real objects and phenomena are so complicated and interrelated that their study and quantitative investigation with due account for all aspects, interrelations, and interactions would present insurmountable mathematical difficulties.

> Reasonable idealization of concrete physics problems constitutes the most important feature of physics as a natural science.

If physicists did not idealize their problems, they could not solve a single concrete problem in full. Very often the simplifying assumptions and limitations are formulated in the problem itself, but at times they are present in the problem in latent or nonexplicit form.

Example 2.1. *A projectile is fired by a gun at an angle* $\alpha = 45°$ *to the horizon with an initial speed* $v_0 = 600$ m/s. *Find the horizontal distance to which the projectile will be propelled. Ignore air drag.*

The problem is formulated. It is idealized. One additional condition simplifying the problem (ignore air drag) is formulated directly. However, many other simplifying conditions are tacitly assumed. It is assumed that

(a) the gun is positioned on Earth;

(b) the motion of Earth about the Sun is not taken into account;

(c) the motion of Earth about its axis is not taken into account;

(d) the vector of acceleration of free fall, **g**, points in the same direction at all points of the projectile's trajectory;

(e) the acceleration of free fall at Earth's surface, g, is assumed constant and equal to 9.8 m/s^2; and

(f) the projectile is thought of as a particle.

Conditions (b)-(e) simplify the problem considerably. Say, if we assume Earth to be a sphere, with the result that the acceleration of free fall points in different directions at different points of the projectile's trajectory, then condition (d) is not met and the problem becomes more complicated. If all the additional assumptions are lifted, the problem becomes extremely complex.

The simplifying assumptions vary from problem to problem, but

> a common feature of all idealizations is the ignoring of nonessential, secondary interrelations and interactions.

Then what should be the criteria? When, in what conditions, can an interrelation or interaction be ignored and when not? The answer is closely linked with the method used in analyzing the solution of a problem and the estimate method, both of which will be discussed in detail later.

Two ways of idealization are most commonly used in analyzing physics problems: the *introduction of idealized physical objects* and the *neglect of nonessential interactions*

and processes. The second approach includes the *introduction of idealized physical processes*.

It is important to note that in any idealized physical object only some of its properties are ignored; the object possesses these properties, but in the specific conditions of the problem they manifest themselves so weakly that they can be ignored.

Physics introduces many idealized objects to use in the idealization of physics problems. Here are some:

(i) *Particle*. A fundamental and universal physical object. In this concept the geometric dimensions of an object are ignored in comparison with the characteristic distances of the problem.

(ii) *Rigid body*. In this idealized object all possible strains are ignored.

(iii) *Elastic body*. Here the remnant strain is ignored. In some problems remnant strain is so small that it plays no role. It must be noted that when elastic bodies interact, there is no transformation of mechanical energy into other types of energy (i.e. the law of energy conservation in mechanics is observed).

(iv) *Inelastic body*. Such an idealized object is incapable of sustaining deformation without permanent change in size or shape. Elastic deformation is ignored.

Examples of other idealized objects will be given later.

The second approach to idealization involves the introduction of idealized physical processes or the neglect of nonessential physical processes (phenomena) and interactions. Examples of idealized processes are the isochoric, isobaric, isothermal, and adiabatic processes.

Very often, the variation of one or another physical quantity is ignored in solving a concrete problem on the assumption that this variation is small. For instance, in the example with the projectile, despite the well-known fact that the acceleration of free fall depends on altitude, we assume that g is constant and equal to $9.8\ \text{m/s}^2$. But since the altitude h reached by the projectile in its flight is small (several kilometers) in comparison with Earth's radius ($R = 6400$ km), it is reasonable to assume that in the given problem the acceleration of free fall, g, is constant.

Thus, as a consequence of idealization and simplification physicists consider a schematic model of the phenomenon rather than a real physical phenomenon.

Usually the *model* of a real physical phenomenon reflects the essentials, allows only for the most important interrelations and interactions, and considers idealized objects rather than real objects. Very often success in solving a particular physics problem depends on how well the model is chosen.

The classification of models of physical phenomena, obviously, coincides with the classification of phenomena proper.

Hence, depending on the properties of the physical system and the conditions in which physical phenomena occur, we distinguish between two general models: classical physical phenomena (the *classical model*) and quantum physical phenomena (the *quantum model*). This book considers only classical physical phenomena and the corresponding models.

3. Classification of Physics Problems

As is known, objects can be classified according to any of their characteristics, but the best classification is according to the *essential characteristics*. Physics problems possess many characteristics. To arrive at an optimal classification we must isolate the *essential features* of a physics problem. So what is a physics problem, what are its essential features? It is also expedient to pose other questions. How and when does a physics problem emerge? What is meant by solving a physics problem? What types of physics problems exist? These questions are not as simple and unimportant as it might seem.

In studying a physical phenomenon some of the physical quantities characterizing the phenomenon may be known while others may not. If a person knows everything about a physical phenomenon (within the framework of the phenomenon), no questions or problems arise. These

arise when in the process of studying a phenomenon the researcher finds that for some reason some of the physical quantities characterizing the phenomenon are unknown. Hence, a problem is posed (stated, formulated) by a person studying a concrete physical phenomenon when some of the interrelations, interactions, physical quantities, etc., in the phenomenon are not known. The following definition of a physics problem can, therefore, be suggested:

> A physics problem constitutes a physical phenomenon, more precisely, a verbal model of this phenomenon (or collection of phenomena), with some known and some unknown physical quantities. To solve a physics problem means to find (reconstruct) the unknown interrelations, physical quantities, etc.

This definition immediately leads us to two classifications of physics problems. The first is based on the difference in the methods of finding the unknown quantities, and the second allows for the content of the physical phenomenon reflected in each physics problem.

There are two ways of finding the unknown quantities in a physical phenomenon: the *experimental* and the *theoretical*. In the first the unknown quantities are determined through measurements performed in experiments. In the second these unknown quantities are found by a physical analysis of the given phenomenon via the physical laws that control the phenomenon. Physical laws interrelate different physical quantities, some of which may be known, others not. If by applying the appropriate physical laws we arrive at a closed system of equations whose unknowns include the unknown physical quantities to be found, solving the system of equations can solve the problem theoretically.

These two ways of finding the unknown quantities lead us to the first classification of physics problems. Problems may be *experimental* and *theoretical*.

> A problem is said to be experimental if its solution requires using measurements.

Experimental problems will not be considered in this book (they are usually studied in laboratory courses).

> A physics problem is said to be theoretical if it corresponds to a physical phenomenon (or collection of phenomena) with some known and some unknown physical quantitities characterizing the phenomenon and can be solved without resorting to measurements.

This book considers only theoretical physical problems. In what follows we drop the adjective "theoretical" for the sake of brevity.

The classification of theoretical problems will be carried out according to the two most important aspects of a physics problem formulated above (a problem is always formulated and solved by a specific person and expresses a certain physical phenomenon). The first aspect divides all physics problems into *formulated* (or *specified*) and *nonspecified*.

> A problem is said to be nonspecified if the data necessary for its solution are not provided in full (except tabulated values) or if the necessary idealization has not been carried out or if both requirements are not fulfilled.

We will consider nonspecified problems later.

> In a formulated problem not only is the complete set of quantities and their values given but also the idealization process has been carried out.

Hence, in a way a formulated problem is a "prepared" problem that always has a solution.

We now turn to the classification of formulated problems according to the second important aspect: a problem expresses a certain physical phenomenon. The same criteria used to classify physical phenomena classify physics problems:

> The type of physics problem corresponds to the type of physical phenomenon that it expresses.

In short, the problem corresponds to the phenomenon-The general criterion mentioned earlier enables us to di.

vide all problems into *classical* and *quantum*. Next, each classical problem (and, of course, each quantum problem) can be referred, via particular criteria, to the corresponding types (right down to the elementary criteria). But it is hardly reasonable to carry out such a detailed classification of problems, not only because this would first require elaborating the entire collection of physical phenomena (i.e. the entire course of general physics) but also because a student beginning the study of physics would find this extremely cumbersome and, apparently, of little use in problem solving. Hence, we restrict our discussion to the above-mentioned general classification according to two generalized features (a formulated and nonspecified, classical and quantum). Note that although an analysis of a physical system enables us to determine, often even before the final solution has been obtained, whether a problem is classical or quantum, what idealized objects and processes have been introduced, what interactions take place, and what their consequences are, it is sometimes only after the problem has been solved that we can decide whether it belongs to the formulated type or the nonspecified.

Another important concept that has proved useful is that of a *basic problem*. Each physical phenomenon can be characterized by a set of physical quantities, which are related by various physical laws. Among the many physical laws that govern a given physical phenomenon there is always one or several fundamental laws.

> Finding the physical quantities that enter into the fundamental laws constitutes the essence of the basic problem of a physical phenomenon.

The next step is to determine, by employing secondary laws, the entire set of physical quantities that characterize the given phenomenon. It can be demonstrated that

> any basic problem in the general physics course consists of finding the state of the corresponding physical system.

Chapter 2

SOME GENERAL METHODS FOR SOLVING PHYSICS PROBLEMS

4. Stages in Solving a Formulated Problem

In solving a formulated problem it is useful to distinguish between three stages: the *physical*, the *mathematical*, and *analysis of solution*.

The physical stage begins with acquainting oneself with the terms (or hypothesis) of the problem and finishes with setting up a closed system of equations whose unknowns include the sought quantities. When the closed system of equations has been set up, the problem is assumed solved from the standpoint of physics.

The mathematical stage begins with solving the closed system of equations and finishes with obtaining a numerical answer. This stage can be broken down into two stages:

(a) finding the solution to the problem in general form; and

(b) finding the numerical answer to the problem.

Once we have solved the system of equations, we have found the solution in general form. Performing the necessary calculations, we arrive at the numerical answer.

In the mathematical stage there is almost no physics. It goes without saying that the mathematical stage is less important than the physical, but it must be stressed that it is not of secondary importance. Unfortunately, the role of this stage is sometimes underestimated. It is sometimes assumed that it can be ignored. It is also wrong to assume that mistakes made in the mathematical stage are secondary. If in solving the system of equations or transferring to new units of measurements or in the calculations proper a mistake is made, the whole solution becomes wrong.

From the practical standpoint, a problem is solved correctly only if correct general and numerical answers are obtained.

Another reason why it is wrong to consider the mathematical stage of secondary importance is that the analysis-of-solution stage must follow. And this stage cannot be carried out if the general and numerical answers have not been obtained. Thus, the final solution to a physics problem relies to an equal degree on both the physical stage and the mathematical stage.

After the solution has been obtained in general form and in the form of a numerical answer, the stage of analysis begins. In this stage we establish how and on what physical quantities the found quantity depends, in what conditions this dependence materializes, etc. In conclusion of the analysis we consider the possibility of formulating and solving other problems obtained by varying and transforming the terms of the given problem. Sometimes the method of dimensionalities is used to check the correctness of the general solution. Bear in mind that this method yields only the necessary indication of the correctness.

Analysis of the numerical answer often involves the following procedures:

(i) study of the dimensions of the obtained quantity;

(ii) verification that the obtained numerical answer is physically meaningful and corresponds to the possible values of the sought quantity; say, if the velocity of an object is found to be higher than the speed of light in empty space ($c = 3 \times 10^8$ m/s), the answer is, obviously, erroneous; and

(iii) investigation of the correspondence of the answers to the terms of the problem if a multivalued answer has been obtained.

To a certain extent analysis of a problem solution is a creative process. Hence, the method just described must not be too rigid and may incorporate a number of other elements, depending on the hypothesis of the problem. Analysis of the solution is closely related to the problem statement method, which will be discussed later.

The system of stages in solving a formulated physics problem is important not in itself. It is necessary to know more than these stages when solving concrete problems. But the special feature of the system of stages is that it

is directly related to the system of methods for solving physics problems. In each stage the person solving the problem must act independently in accordance with the particular stage.

> It is often said, in fact, that if you want to learn to solve physics problems, you must solve them independently.

This is true, of course. But if the person solving the problem is not taught the general ways (methods) of problem solving, he will use the torturous trial-and-error method. This makes it necessary to have a *system of general methods* for carrying out all the stages in the solution of an arbitrary physics problem, a system that serves as a set of rules for independent work. Hence, the system of general methods must have the following properties:

(α) it must be universal, that is, applicable to the solution of any problem in the general physics course; and

(β) it must encompass all the stages of solution.

After analyzing the ways in which each stage in problem solving needs to be carried out, we suggest the following system of general methods:

(1) analysis of the physical content of a problem;

(2) application of a physical law;

(3) general-particular methods;

(4) simplification and complication method; and estimate;

(5) analysis of the solution; and

(6) problem statement.

It must be noted that no method taken alone is universal. Each has meaning and manifests itself fully only within the system of methods, and this system does not always automatically guarantee solution of the problem. Sometimes a problem can be solved without applying any method, that is by intuition. But solutions will be obtained much faster and more often if one acts in accordance with these methods. In short,

> the system of general methods is not a dogma but a guide to independent work in solving a physics problem, not an instruction but a system of intelligent advice.

There are appropriate methods for realizing each stage in the problem solving process. The following sections discuss each method in detail.

5. Method of Analyzing the Physical Content of a Problem

The solution of any physics problem is primarily a mental process. However, we have no wish to go into the psychological aspects of this process. Let us go directly to the results.

Any physics problem expresses a physical phenomenon or group of phenomena. The relationships between the sought and known physical processes lie inside this phenomenon. To find these relationships, which must form a closed system of equations, one must not only know the essence of the given phenomenon, the system of its physical parameters, the laws that govern the system, and the limits of its applicability, but also how to isolate all these elements in the given problem.

> Practically speaking, analysis of the physics of a problem is reduced to isolating and analyzing the physical phenomenon.

But how does one begin to analyze the physical content of a problem?

The *introductory part* of the method of analyzing the physical content of a problem is of an auxiliary nature: a kind of entry into the world of the physical phenomena inherent in the problem. Analysis of the phenomena takes place when the student is getting acquainted with the problem. After reading the problem, it is advisable to write down its terms and try to understand the data and sought quantities and the relationship between them. Next it is necessary to make a sketch, drawing, or diagram and mark all the data and quantities on it. A graphical image makes it possible to picture the physical phenomena contained in the problem.

In the *main part* of the method we make a concrete analysis of the physical phenomena. As is known, a physical phenomenon always has two sides, the *qualitative*

and the *quantitative*. Hence, we start the analysis by establishing the qualitative nature of the phenomenon (in what respects the phenomenon differs from other phenomena, what its essence is, how it occurs, and so on). Specifically, we, first, select the appropriate physical system (what physical objects must be included in the system); second, determine the qualitative characteristics of these objects (the type of idealized object that each component in the system represents: particle, rigid body, etc.); and, third, determine in which physical processes the components of the system participate.

After this we establish the various quantitative interrelations between the various physical quantities characterizing the given phenomenon. As noted earlier, the quantitative interrelations between various physical quantities are reflected in physical laws. Hence, employing the appropriate physical laws, we arrive at a closed system of equations. Setting up such a system completes the physical solution of the problem.

Thus, the method of analyzing the physical content of a problem indicates how to start solving a problem and how and what must be done to solve any formulated physics problem. Understandably, this method can be ap. plied only at the physical stage of problem solving-

6. General-Particular Methods. The DI Method

The system of general-particular methods is universal in the sense that it can be applied to solving problems in practically all parts of the general physics course. Once one has mastered a fairly small number of general-particular methods, one can successfully solve practically any formulated problems.

There are relatively few general-particular methods. Among these we will consider the following: the *kinematic*, the *dynamical*, the *conservation-law*, the *method of calculating physical fields*, and the *method of differentiation and integration*. The first four will be studied in subsequent chapters. Here we examine the last method, the DI method.

Of great importance to the DI method is the principle

of the *limits of applicability* of physical laws. As is known, the content of a physical law is not absolute, and the validity of a law is restricted to the framework of the applicability limits (or conditions).

Often a physical law can be expanded by changing its form beyond the limits of applicability via the DI method. Two principles underlie this method: *the principle*

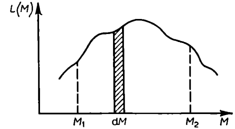

Figure 6.1

that a law can be represented in differential form, and *the superposition principle* (if the quantities entering into the law are additive).

The essence of the DI method lies in the following. Suppose that the physical law in question has the form

$$K = LM, \tag{6.1}$$

where K, L, and M are physical quantities, with $L =$ const being the condition of the law's applicability. How should one generalize this law to the case where L is not a constant but a function of M, that is, $L = L(M)$?

Let us isolate an interval $\mathrm{d}M$ so small that the variation of M over this interval can be ignored (Figure 6.1). Thus, in the interval $\mathrm{d}M$ we can approximately assume that L is constant ($L =$ const) and, hence, the law (6.1) is valid in this interval (approximately). Then

$$\mathrm{d}K = L(M)\,\mathrm{d}M, \tag{6.2}$$

where $\mathrm{d}K$ is the variation of K over $\mathrm{d}M$.

Employing the superposition principle (summing the quantities (6.2) over all the intervals of variation of M), we arrive at an expression for K in the form

$$K = \int_{M_1}^{M_2} L(M)\,dM, \tag{6.3}$$

where M_1 and M_2 are the initial and final values of M.

Thus, the DI method consists of two parts. In the first we *find the differential* (6.2) of the sought quantity. This is done in the majority of cases by partitioning the object (mentally) into parts so small that they can be considered as being particles or by partitioning a large time interval into time intervals dt so small that in the course of each interval the process being investigated may be thought of as approximately uniform (or steady-state).

In the second part of the method we *sum* (or *integrate*). The most difficult aspect here is to *choose the correct integration variable* and *determine the limits of integration*. To determine the integration variable we must analyze in detail on what quantities the differential of the sought quantity depends and what variable is the most important. This variable is usually chosen as the integration variable, and the other variables are expressed as functions of the integration variable. As a result the differential of the sought quantity assumes the form of a function of the integration variable. The next step is to determine the limits of integration, which are the upper and lower values of the integration variable. Evaluation of the definite integral yields the numerical value of the sought quantity.

Example 6.1. *A thin rod of length $l = 1$ m has been uniformly charged with an electric charge $Q = 10^{-12}$ C. Determine the potential of the electric field generated by this charge at a point A on the rod's axis at a distance $d = 1$ m from its end (Figure 6.2). The rod is placed in a vacuum.*

Solution. An answer written in the form $\varphi = Q/4\pi\varepsilon_0 d$, from which it follows that $\varphi = 9 \times 10^{-3}$ V, is erroneous since this formula is valid only for the potential of an electric field generated by a point charge. In our case,

however, charge Q is distributed over an object (the rod) whose dimensions ($l = 1$ m) cannot be ignored in comparison with the characteristic distance ($d = 1$ m) considered in the problem. Hence, charge Q cannot be considered a point charge.

Let us employ the DI method. We partition the rod into so many small segments that each can be taken as a particle. Hence, the charge carried by each segment can be taken as a point charge. We now take one segment of

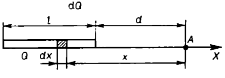

Figure 6.2

length dx that lies at a distance x from point A. The charge of this segment is a point charge and amounts to $dQ = (Q/l)\ dx$. Charge dQ generates an electric field whose potential $d\varphi$ at point A can be calculated according to the following formula:

$$d\varphi = \frac{dQ}{4\pi\varepsilon_0 x}. \tag{6.4}$$

Substituting the value $dQ = (Q/l)\ dx$, we arrive at a formula that expresses the sought quantity as a function of a single variable:

$$d\varphi = \frac{Q}{4\pi\varepsilon_0 l}\ \frac{dx}{x}. \tag{6.5}$$

The first part of the method is complete. Now we must sum the potentials of the fields generated by all the elementary charges (all are point charges by construction) into which the initial charge Q has been divided. The variable of integration x varies from $d = 1$ m to $d + l = 2$ m. Integrating (6.5) with respect to x within these limits, we arrive at the following formula for calculating the sought quantity:

$$\varphi = \int_{d}^{d+l} \frac{Q\, dx}{4\pi\varepsilon_0 l x} = \frac{Q}{4\pi\varepsilon_0 l} \ln\left(1 + \frac{l}{d}\right).$$

Substituting the required numerical values, we find that $\varphi \approx 6.3 \times 10^{-3}$V.

The differentiation and integration method is universal and is needed both in the study of theory and especially in solving physics problems. In mechanics the method is used to calculate the work performed by a variable force and the moments of inertia; in the study of physical fields it is used to calculate the field strengths and potentials of fields generated by extended masses, nonpoint-like charges, macrocurrents, etc.

The differentiation and integration of functions forms the mathematical basis of this method. Hence, the method makes it possible to trace the links existing between physics and higher mathematics (calculus) when studying these courses.

7. The Simplification and Complication Method. The Estimate Method

The combination of these two methods is used in solving complicated problems and also nonspecified and nonstandard problems. It is widely used to analyze the solution of a physics problem. At this stage the simplification and complication method makes it possible to develop practically any problem into a "block" of more complicated or simpler problems. A typical example is Example 11.2 (see Section 11).

The component parts of the simplification and complication method are two interconnected and opposite processes: the *simplification* (idealization, estimate and discarding of secondary phenomena, neglect of unessential elements, etc.) and *complication* (consideration of previously discarded objects, phenomena, and elements, and complication of the physical system, interrelations, etc.). The *estimate method* forms the material basis of these two processes.

This approach is often employed in analyzing a physical situation by *estimating physical quantities* or *physical phenomena*.

An estimate of a physical quantity consists, first, in numerically calculating the order of magnitude of the

physical quantity proper (*the order-of-magnitude estimate*) and, second, in comparing similar quantities by their order of magnitude (*comparison of the orders of magnitude*).

When a quantity depending on other quantities is calculated, the numerical value of each of the component quantities is represented in the scientific notation (in this notation numbers are expressed as products consisting of a number between 1 and 10 multiplied by an appropriate power of 10). Then the order of magnitude of each term (if the calculated expression is an algebraic sum) is estimated. The terms with the greatest order are isolated, while the terms whose orders are lower than the highest-order terms by at least two are discarded. The exact significant digit can be found by using a calculator.

Example 7.1. *Suppose that the general solution of a problem gives the following working equation:*

$$\Delta m = \frac{VM(p_1T_2 - p_2T_1)}{RT_1T_2},$$

where $V = 9$ l *is the volume occupied by a gas,* $M = 2 \times 10^{-3}$ kg/mol *the molecular weight of the gas,* $p_1 = 52 \times 10^5$ Pa *the initial gas pressure,* $T_1 = 296$ K *the initial gas temperature,* $p_2 = 5 \times 10^4$ Pa *the final gas pressure,* $T_2 = 283$ K *the final gas temperature,* $R = 8.31$ J/mol·K *the universal gas constant, and* Δm *the variation of the gas mass. Estimate the order of magnitude of* Δm.

Solution. We transfer the data into the SI system and, simultaneously, round off their values and represent these values in the scientific notation. We have: $V \approx 10^{-2}$ m^3, $M = 2 \times 10^{-3}$ kg/mol, $p_1 \approx 5 \times 10^6$ Pa, $T_1 \approx 3 \times 10^2$ K, $p_2 = 5 \times 10^4$ Pa, $T_2 = 3 \times 10^2$ K, and $R \approx 8$ J/mol·K. These data imply, first, that the approximate values of the initial and final gas temperatures coincide and, hence, instead of the above formula we arrive at a simpler expression:

$$\Delta m \approx \frac{VM(p_1 - p_2)}{RT}.$$

Second, the order of magnitude of the final pressure $p_2 = 5 \times 10^4$ Pa is much lower than that of the initial pressure $p_1 = 5 \times 10^6$ Pa (by two orders of magnitude) and, hence, the final pressure can be ignored. The final formula for the estimate of the order of magnitude of Δm is $\Delta m \approx VMp_1/RT$, whence

$$\Delta m = \frac{10^{-2} \times 2 \times 10^{-3} \times 5 \times 10^6}{8 \times 3 \times 10^2} \text{ kg} \approx 4 \times 10^{-2} \text{ kg}.$$

A more exact but more cumbersome calculation yields the following value of the sought quantity: $\Delta m = 3.8 \times 10^{-2}$ kg.

A rough but quick estimate of the order of magnitude of a sought quantity is extremely important for the subsequent analysis of the solution.

In comparing physical quantities that depend on other quantities the first step is to find their ratio in general form and the second to numerically calculate the order of magnitude of this ratio.

Example 7.2. *Compare the force of gravity F_g existing between two protons and the force of repulsion due to the proton electric charge, F_{el}.*

Solution. Let us find the ratio of the two forces:

$$\frac{F_g}{F_{el}} = \frac{Gm^2 \times 4\pi\varepsilon_0 r^2}{r^2 \times Q^2},$$

where $G \approx 6.7 \times 10^{-11}$ N·m²/kg² is the universal gravitational constant, $m \approx 1.67 \times 10^{-27}$ kg the proton mass, $Q = 1.6 \times 10^{-19}$ C the proton charge, and $4\pi\varepsilon_0 \approx 1.1 \times 10^{-10}$ F/m. Calculation of this ratio yields $F_g/F_{el} \approx 7 \times 10^{-37} \approx 10^{-36}$. We see that the force of gravity existing between two protons is weaker than the force of electric repulsion by 36 orders of magnitude (the gravitational interaction is fantastically weak if compared with the electromagnetic interaction).

Example 7.3. *Which body attracts the Moon with greater strength, Earth or the Sun?*

Solution. On the basis of Newton's law of gravitation we can find the ratio of the force of gravity exerted on the

Moon by Earth, F_E, to the force of gravity exerted on the Moon by the Sun, F_S:

$$\frac{F_E}{F_S} = \frac{M_E r_S^2}{M_S r_E^2},$$

where $M_E \approx 6 \times 10^{24}$ kg is Earth's mass, $M_S \approx 2 \times 10^{30}$ kg the Sun's mass, $r_S \approx 1.5 \times 10^{11}$ m the mean distance from the Moon (Earth) to the Sun, and $r_E = 4 \times 10^8$ m the mean distance from the Moon to Earth. The calculation yields $F_E/F_S \approx 3/8$. Hence, by order of magnitude the forces of attraction of the Moon to Earth and the Sun are equal, but, nevertheless, the Sun attracts the Moon about two and a half times as strongly as Earth attracts the Moon. There is nothing paradoxical in this if one recalls that under the Sun's attraction the Moon moves around the Sun while under Earth's attraction the Moon moves around Earth.

> Estimate of a physical phenomenon amounts, first, to finding the fundamental law governing the phenomenon and, second, to numerically calculating the order of magnitude of the respective physical quantity.

Estimate problems are often nonspecified.

Example 7.4. *Estimate the pressure at Earth's center.*

Solution. Statement of the problem. We introduce some simplifying assumptions. We will think of Earth as a homogeneous solid ball of radius R_E. The field of gravity generated by a homogeneous ball is equivalent to the field of gravity generated by a particle of the same mass as the ball's and placed at the center of the ball. Any object of mass m placed on Earth's surface is attracted to Earth with a force $F_g = GmM_E/R_E^2$ and, hence, presses on the surface with a pressure $p = F_g/A$, where A is the surface area of contact of object and Earth. If many such objects are distributed over Earth's surface in the form of a thin spherical layer, the pressure of this layer of mass dm on Earth's surface will be

$$\mathrm{d}p = \frac{GM_E\,\mathrm{d}m}{4\pi R_E^4}.$$

The force of attraction to Earth depends on the distance to Earth's center. Hence, the thickness of the spherical layer must be small in comparison to this distance. Each spherical layer presses on the layers lying under it. It is now clear that to calculate the pressure at Earth's center we must employ the DI method (see Section 6). We partition the solid spherical ball representing Earth into thin spherical layers and consider a layer of thickness dr lying at a distance r from Earth's center O (Figure 7.1).

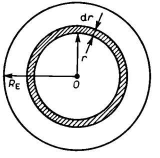

Figure 7.1

This layer is attracted to the part of Earth lying within it (the outer part does not act on the layer) with a force

$$dF_g = \frac{G \times 4\pi r^2\, dr\rho \times 4\pi r^3 \rho}{3r^2},$$

where ρ is Earth's mean density. This yields the following formula for the pressure produced by the layer:

$$dp = \frac{dF_g}{4\pi r^2} = \frac{4\pi G\rho^2 r\, dr}{3}.$$

After integration we get

$$p = \int_r^{R_E} dp = \frac{2\pi}{3} G\rho^2 (R_E^2 - r^2),$$

which is the pressure inside Earth at a distance r from Earth's center. At $r = 0$ we have the pressure at Earth's center:

$$p = \frac{2\pi}{3} G\rho^2 R_E^2.$$

We estimate the order of magnitude of this quantity (assuming that $\rho \approx 5.5 \times 10^3$ kg/m^3):

$$p \approx 1.6 \times 10^{11}\ \text{Pa} \approx 2 \times 10^{11}\ \text{Pa}.$$

As is known, the standard atmospheric pressure amounts to about 10^5 Pa. Hence, the pressure at Earth's center exceeds the standard atmospheric pressure by six orders of magnitude.

> The estimate method is often used to compare different physical phenomena. Here the fundamental physical quantities characterizing the phenomena are estimated.

Example 7.5. *A flat conducting contour encompassing an area $A = 1$ m^2 has an electrical resistance $R = 1\Omega$. It is placed in a homogeneous magnetic field whose induction varies according to the law $B = B_0 - \alpha t^2/2A$, with $B_0 = 10$ T and $\alpha = 10^{-1}$ T·m^2/s^2. The plane in which the contour lies is perpendicular to the magnetic induction vector* **B**. *Determine the current flowing in the contour at time $t = 1$ s if the inductance of the contour is L and if at $t = 0$ the current in the contour is $I = 0$.*

Solution. Depending on the value of L the concrete physical phenomena will occur in different ways. We consider two limiting cases.

(1) Inductance L is so small that self-induction can be ignored. But ignored in comparison with what other physical phenomenon (or quantity)? We incorporate the contour and the magnetic field into the physical system. Because of the variation of the magnetic field, electromagnetic induction sets in. Since the induction emf $\mathscr{E}_i = \alpha t$ is time-dependent, the induction current is also time-dependent. Hence, the phenomenon of self-induction sets in, with the self-induction emf $\mathscr{E}_{s-i} = -L(\mathrm{d}I/\mathrm{d}t)$ acting as a characteristic of this phenomenon.

Thus, we are considering the case of L-values so small that the self-induction emf $\mathscr{E}_{s-i}$ can be ignored in comparison with the induction emf $\mathscr{E}_i$. Then Kirchhoff's second rule yields $\mathscr{E}_i = I_i R$. Since $\mathscr{E}_i = \alpha t$, we get

$$I_i = \alpha t/R, \quad I_i = 10^{-1}\text{A}. \qquad (7.1)$$

(2) Inductance L is so large that we cannot ignore self-induction. This means that the self-induction emf $\mathscr{E}_{s-i}$ is comparable to the induction emf $\mathscr{E}_i$. Kirchhoff's

second rule then yields

$$\mathscr{E}_1 - L\frac{\mathrm{d}I}{\mathrm{d}t} = IR, \quad \text{or} \quad \alpha t - L\frac{\mathrm{d}I}{\mathrm{d}t} = IR.$$

The solution to this differential equation that satisfies the initial condition ($I = 0$ at $t = 0$) is given by the following function:

$$I = \frac{\alpha t}{R} - \frac{\alpha L}{R^2}\left(1 - \mathrm{e}^{-(R/L)t}\right). \tag{7.2}$$

The second term on the right-hand side accounts for self-induction. Suppose that inductance L of the contour is equal to 1 H. Substituting the necessary values into (7.2), we get $I = 0.04$ A, which differs considerably from the value obtained earlier. We see that for large values of the inductance of the contour one cannot ignore self-induction.

Note that the above conclusion is valid only for short time intervals. Equation (7.2) clearly shows that the role of self-induction diminishes with the passage of time. For example, at $t = 100$ s we have $I = 10$ A if self-induction is ignored and $I = 9.9$ A if it is not, that is, the correction introduced by self-induction amounts to only 1 percent. Thus, for time intervals exceeding 100 s we can ignore self-induction even for such a large value of the contour's inductance ($L = 1$ H).

8. The Problem Statement Method

This method is employed either when the solution is being analyzed or, more often, when the problem is being formulated in the case of a nonspecified problem.

The reader will recall that a nonspecified problem has been defined as a problem that has yet to be idealized or a problem with an incomplete (nonclosed)* system of physical quantities and conditions or as a problem in which both conditions are present. Hence, a nonspecified

* Tabulated data are not considered as fixed quantities. Thus, an idealized problem whose system of fixed (or given) quantities is incomplete only because it lacks tabulated data does not constitute a nonspecified problem.

problem differs from a formulated one in that, first, it is not idealized and, second, its solution is not unique, with the result that the problem breaks down into several formulated problems.

Often a nonspecified problem contains no concrete data or it is not known what quantities must be found or some additional conditions are missing, and so on.

> Hence, the first step (the most important but also the most difficult) in solving a nonspecified problem is to state (formulate, specify) the problem properly.

When analyzing a physical phenomenon (the problem statement method starts with this stage), one must establish what simplifying assumptions can be introduced, what can be ignored, what additional conditions can be introduced, and the like. Above, this procedure was called the idealization process. After reasonable idealization, it is necessary to establish what data may be known, and what can be taken from handbooks, tables, and other sources. Some data may prove superfluous, while other data may prove lacking, but this can only be established after the problem has been solved in general form. Apparently, there is no general method (or algorithm) for carrying out the idealization process in a problem. This is simply a creative process.

Idealization is followed by formulation (or statement) of the problem: under such and such conditions something is given and something must be found. This finishes the first stage in the solution and statement of a nonspecified problem. The problem has been stated (formulated). The next stage is already known: the solution of a formulated problem. We must again analyze the physical phenomenon in question (now this is done much faster), set up the closed system of equations, and solve the system in general form. Before proceeding with the numerical calculations, we must check to see whether all the necessary data for this are present. If not, the lacking data must be added to the given data or taken from tables, handbooks, and other sources. Only after introducing the additional data that guarantee the existence of a unique solution to the formulated problem can we assume

that the problem has been properly stated (formulated). Finally, comes the arithmetic to yield the total solution.

Next, lifting one or several additional conditions (say, by allowing for friction or by assuming that the given object is not a particle), we can formulate other problems and solve them in the above manner.

Thus, a single nonspecified problem may be linked with a whole group, or "block", of different physics problems of varying difficulty.

Example 8.1. *An object is placed on a wedge (an inclined plane). Study the motion of the wedge and the object (Figure 8.1).*

Solution. This is a nonspecified problem. It is not clear what physical quantities are assumed given or what must be found, and there are no additional conditions (where must one look for the data on the object, what are their properties, and so on).

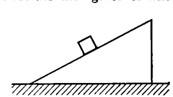

Figure 8.1

In the first stage of analyzing a possible physical phenomenon let us try to formulate the problem properly. It is advisable to include the object and the wedge in the physical system and assume that all other objects are external to the system.

Let us now idealize the problem. To this end we introduce additional conditions and limitations which will ensure that the problem yet to be formulated will have a solution. We assume that

(1) the given physical system is placed on Earth;

(2) the friction between the wedge and Earth's surface is so large that the wedge remains fixed in relation to Earth;

(3) both object and wedge are rigid bodies, that is, all possible strains are so small that they can be discarded; however, the elastic forces generated by these strains will be accounted for (this condition, for one thing, implies that the surfaces of the wedge can be considered flat);

(4) the height of the wedge is so small that over its entire value we can consider $g = 9.8\ \text{m/s}^2 = \text{const}$;

(5) the object is a particle;

(6) the friction between the object and the wedge is small and can be discarded;

(7) the horizontal face of the wedge is so short that the sphericity of Earth can be ignored (i.e. the acceleration of free fall **g** can be assumed to point everywhere in the same direction).

Now, introducing these conditions and limitations, we can formulate (state) the first problem:

A particle of mass $m = 1$ kg moves along a rigid inclined plane whose height is $h = 10$ m. The initial velocity of the object is $v_0 = 0$. The angle at the base of the inclined plane is $\alpha = 30°$. Determine the time it takes the object to slip down the plane to the base (or the object's acceleration a or the velocity v or some other parameter) if there is no friction between object and wedge. Air resistance is ignored.

The problem has been formulated and, as its solution will shortly show (the solution is quite simple), has been formulated correctly. Analysis of this solution shows that the sought time t depends on height h and angle α in the following manner:

$$t = \frac{1}{g \sin \alpha} \sqrt{2gh}. \tag{8.1}$$

Substitution of the numerical values into this formula yields $t \approx 3$ s.

One formulated problem has been solved. Now, by gradually lifting the restrictions and additional conditions, more complicated problems can be formulated. Say, if we lift condition (6), we get a problem on the motion of a particle that allows for friction. It is advisable to compare the solution of this second problem with the first solution (8.1). If we lift condition (5), we have a problem on a rigid body (not a particle) moving along an inclined plane. But here we must introduce the assumption concerning the shape of the object (a ball, cylinder, or some other shape). The solution to the third problem can be compared with the solutions to the first and second and the emerging differences can be analyzed (say, why t is different for the three cases).

Thus, from a single nonspecified problem we can obtain a whole set ("cluster") of diversified problems.

9. Another Classification of Formulated Problems

Another classification of formulated problems has also proved useful. It is based on an extremely important aspect of the process of problem solution, namely, the means necessary and sufficient for solving a physics problem.
According to this, formulated problems can be classified as *elementary*, *standard*, or *nonstandard problems*. Here we give definitions of such problems with little attention paid to rigor, although more rigorous definitions could also be given.

An elementary problem is one for whose solution it is necessary and sufficient to recall and use only one physical law.
A standard problem is a formulated problem for whose solution it is necessary and sufficient to employ only a system of "common" knowledge and "standard" methods.

The widespread collections of physics problems usually contain standard problems. Below we give examples of elementary, standard, and nonstandard problems.

Example 9.1. *A direct current $I = 1$ A flows in a conductor made in the form of a circle of radius $R = 0.5$ m. Determine the magnetic induction generated by this current at the center of the circle. The conductor is placed in a vacuum.*

Solution. The solution is clear-cut. We need only to write the Biot-Savart law in integral form for a circular current:

$$B = \mu_0 \mu \frac{I}{2R}, \quad \text{or} \quad B = 4\pi \times 10^{-7} \text{ T}.$$

Thus, to solve this problem it is necessary and sufficient to employ a concrete law, and the method of applying this law depends on the form in which the law is writ-

ten. Consequently, this problem is elementary. Sometimes elementary problems are called *training* or *plug-in* problems. Problems of this type do justify this classification. They can be called training because they train the student's memory; they can be called plug-in because after writing the necessary law the student need only substitute (or "plug in") the various values of the quantities and carry out the calculations; they can also be called elementary. We will reserve the last name for such

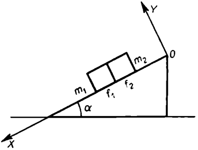

Figure 9.1

problems. Bearing in mind that elementary problems can be solved without employing the general approach (although some elements of this approach can be used in solving such problems), we do not consider them in this book.

Example 9.2. *Two blocks of masses m_1 and m_2 are placed on a plane inclined at an angle α to the horizontal (Figure 9.1). Determine the force of interaction between the blocks (which are touching each other) during their motion if the coefficients of friction between the inclined plane and the blocks are f_1 and f_2, respectively, with $f_1 > f_2$.*

Solution. This fairly simple problem cannot be solved by simply writing the appropriate physical law (say, Newton's second law $\mathbf{F} = m\mathbf{a}$) if only because we must know not only the law but also how to apply it.

Let us employ the method of analyzing the physical content. After writing the hypothesis of the problem, plot-

ting the necessary diagram, and analyzing the data and the sought quantities, we get down to the main part of the physical analysis. We incorporate in the physical system the blocks m_1 and m_2 and assume that all other objects are external to this system. The objects of the system may be taken to be particles. These particles move within the system owing to their interaction with each other and with external objects (Earth and the inclined plane). We must determine one parameter of such interaction, that is, one of the internal forces. This problem is linked with the basic problem of particle dynamics. Let us apply Newton's second law to each object (particle). We associate an inertial reference frame with the inclined plane and point the axes of the coordinates as shown in Figure 9.1. Clearly, there are four forces acting on each of the two objects m_1 and m_2, namely, the force of gravity mg, the force $\mathbf{N}$ exerted on a block by the plane, the friction force $\mathbf{F}_{fr}$, and the sought force of interaction between the blocks, $\mathbf{F}$. Projecting these forces on the coordinate axes, we get a closed system of two equations in two unknowns:

$$m_1 g \sin \alpha - f_1 m_1 g \cos \alpha + F = m_1 a,$$
$$m_2 g \sin \alpha - f_2 m_2 g \cos \alpha - F = m_2 a.$$

Solving this system, we arrive at an answer in general form:

$$F = \frac{m_1 m_2 (f_1 - f_2) \cos \alpha}{m_1 + m_2}.$$

We see that to solve this problem it is necessary and sufficient to use only Newton's second law, the standard method of analyzing the physical content of a problem, and the method of applying the necessary physical law. Hence the solved problem is a standard one.

A nonstandard problem is also a formulated problem, but the application of only "ordinary" laws and methods in the process of its solution does not lead to the goal because the system of equations proves not closed.

Something remains unaccounted for (which makes the problem nonstandard), some feature that eludes imme-

diate detection and requires a wild guess. Obviously, there can be no general or universal practical advice on how to guess this feature.

Example 9.3. *Two particles of masses m_1 and m_2 (with $m_1 > m_2$) are connected by a massless and nonexpandable string as shown in Figure 9.2. The pulleys are also massless. Find the force of tensile strength of the string that emerges when the particles move.*

Solution. We employ the method of analyzing the physical content. After writing out the hypothesis of the problem, plotting the necessary diagram, and analyzing the data and sought quantities, we get down to the second part of the physical analysis. The objects system incorporates the objects m_1 and m_2 and the string. By the hypothesis, objects m_1 and m_2 are particles and the thread is massless and nonexpandable and thus cannot be thought of as a particle. As a result of the interaction of the objects in the system between themselves and with external objects (such as Earth) the objects m_1 and m_2 move in a straight line with accelerations a_1 and a_2, respectively. One of the dynamical parameters of the problem must be found: the force of tensile strength (or simply tensile strength) of the string. This problem is related to the basic problem of particle dynamics. We apply Newton's second law to m_1 and m_2:

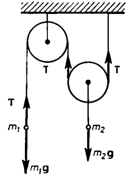

Figure 9.2

$$m_1g - T = m_1a_1, \quad 2T - m_2g = m_2a_2,$$

with T the tensile strength.

Now we have a closed system of two equations in three unknowns (a_1, a_2, and T). The concrete laws of dynamics have been exhausted, so we employ the concrete laws of kinematics:

$$s_1 = a_1t^2/2, \quad s_2 = a_2t^2/2.$$

We have thus arrived at an open system of four equations in six unknowns (a_1, a_2, T, s_1, s_2, and t). Now the concrete laws of kinematics have been exhausted, but the problem still remains unsolved. Something in the hypothesis of the problem has not been accounted for, but we can only guess what this "something" is. We again analyze the hypothesis. Why are the accelerations a_1 and a_2 different? The conditions in which the two particles move are different. But in what way? Well, different forces act on m_1 and m_2 (this is reflected in the dynamics of the problem). In what other respect is their motion different? In the kinematics of the motion; concretely, s_1 differs from s_2. But why? Because a_1 differs from a_2. But this is what we started with. So far logic has proved of no help. And then suddenly, like a bolt out of the blue, we have the answer: $s_1 = 2s_2$. Why? Well, the answer is simple. The guess is indeed correct, and it can be rigorously proved. After this the solution is indeed obvious.

To conclude this section we examine a particular case of nonstandard problems, which we will call *original*, or *challenging*, problems.

> An original problem is a nonstandard problem in the solution of which the role of the wild guess becomes dominant in comparison with ordinary knowledge and methods.

The role of the latter is usually minor in solving such problems. The definitions of an original and a strictly nonstandard problem show that the boundary between the two is hazy. Sometimes in original problems the undefinable "something," or the special feature that evades detection, becomes the seed out of which new nonstandard methods of problem solving grow. Note that an original problem can often be solved by standard methods, but this is so cumbersome, at times involving complicated transformations and calculations, that it proves best to look for another, i.e. original, solution.

Example 9.4. *Two motorboats simultaneously leave two ports A and B separated by a distance l. One has a velocity* v_1 *and the other, a velocity* v_2 *(Figure 9.3). The direction in which the first motorboat travels forms an angle* α *with the*

AB line and that in which the second motorboat travels an angle β with AB. What will be the smallest distance between the motorboats?

Solution. We start with a standard solution. We employ the method of analyzing the physical content. In what follows this method will be called the *method of analysis* for the sake of brevity. The physical system is formed by the two motorboats, which are assumed to be particles.

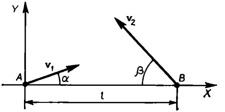

Figure 9.3

These move uniformly and rectilinearly with respect to the inertial reference frame in which Earth is at rest. This motion is considered only formally. We wish to find one of the parameters of this motion, the minimal distance between the particles. The problem is associated with the basic problem of kinematics. We assume the origin to be at point A. Since the laws of motion of the particles are known,

$$\mathbf{r}_1 = v_1 t \cos \alpha \times \mathbf{i} + v_1 t \sin \alpha \times \mathbf{j},$$
$$\mathbf{r}_2 = (l - v_2 t \cos \beta)\ \mathbf{i} + v_2 t \sin \beta \times \mathbf{j},$$

we can find the distance between the motorboats at any moment in time:

$$r = \sqrt{[\,l - (v_2 \cos \beta + v_1 \cos \alpha)\,t]^2 + [(v_2 \sin \beta - v_1 \sin \alpha)\,t]^2}. \tag{9.1}$$

There only remains to find the minimum of this expression. Here, however, we will encounter some difficult calculations, but these will have to be carried out to the end. To simplify the calculations (we must find the de-

rivative r' and, nullifying it, find the value t_{min}, which is then substituted into (9.1), and the sought value of r_{min} is found) we square r and get

$$r^2 = [l - (v_2 \cos\beta + v_1 \cos\alpha)\, t]^2 + [(v_2 \sin\beta - v_1 \sin\alpha)\, t]^2.$$

We find the derivatives of both sides of this equation:

$$2rr' = 2\{-[l - (v_2 \cos\beta + v_1 \cos\alpha)\, t]\,(v_2 \cos\beta + v_1 \cos\alpha) + (v_2 \sin\beta - v_1 \sin\alpha)^2 t\}.$$

We exclude the trivial case where $r = 0$ (this would mean that the motorboats would eventually collide). Then, equating r' with zero, we find the time t_{min} at which the distance between the motorboats is minimal:

$$t_{min} = \frac{l\,(v_2 \cos\beta + v_1 \cos\alpha)}{v_2^2 + v_1^2 + 2v_1 v_2 \cos(\alpha+\beta)}.$$

Substituting this expression for t_{min} into (9.1) and performing tedious calculations (we advise the reader to go through these calculations independently), we arrive at the following final expression:

$$r_{min} = \frac{l\,(v_2 \sin\beta - v_1 \sin\alpha)}{\sqrt{v_1^2 + v_2^2 + 2v_1 v_2 \cos(\alpha+\beta)}}.$$

We now give the original solution. We fix the inertial reference frame with the first motorboat instead of Earth. But why? In that respect is the new frame better than the old frame connected with Earth? Perhaps it is better, perhaps not. We cannot know in advance. But still let us select such a reference frame. Then the second motorboat moves in relation to this reference frame with a relative velocity

$$\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 \tag{9.2}$$

and its trajectory is the straight line BC (Figure 9.4). Obviously, the minimal distance between the motorboats is equal to the length of the perpendicular AC dropped from point A onto the straight line BC:

$$|AC| = l \sin\varphi,$$

where φ is the angle between BA and vector **v**. What remains to be found is $\sin\varphi$. Projecting **v** (see Eq. (9.2))

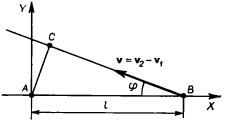

Figure 9.4

on the Y axis, we get

$$v \sin\varphi = v_2 \sin\beta - v_1 \sin\alpha.$$

By the law of cosines,

$$v = \sqrt{v_1^2 + v_2^2 + 2v_1v_2 \cos(\alpha+\beta)}.$$

Thus,

$$\sin\varphi = \frac{v_2 \sin\beta - v_1 \sin\alpha}{\sqrt{v_1^2 + v_2^2 + 2v_1v_2 \cos(\alpha+\beta)}}.$$

Hence, finally,

$$r_{\min} = |AC| = \frac{l(v_2 \sin\beta - v_1 \sin\alpha)}{\sqrt{v_1^2 + v_2^2 + 2v_1v_2 \cos(\alpha+\beta)}},$$

which coincides with the expression obtained via tedious calculations by the standard method.

Part 2

SOLUTION OF STANDARD PROBLEMS

MECHANICS

Chapter 3

THE MOTION OF A PARTICLE

10. Particle Kinematics

In kinematics the motion of objects is considered in a formal manner, without explaining the reasons for the variations in motion and, hence, without employing the concepts of force or mass.

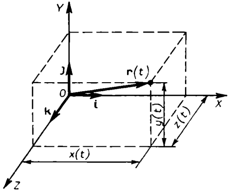

Figure 10.1

The simplest physical system consists of a single particle or a relatively small number. The position of a particle with respect to a reference frame at an arbitrary moment in time t is determined by the radius vector $\mathbf{r} = \mathbf{r}(t)$ (Figure 10.1). If we introduce the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ along the X, Y, and Z axes, respectively, the radi-

us vector $\mathbf{r}$ can be represented in the following form:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \qquad (10.1)$$

with $x(t)$, $y(t)$, and $z(t)$ the components of $\mathbf{r}(t)$. Simultaneously specifying three functions, $x(t)$, $y(t)$, and $z(t)$, is equivalent to specifying a single vector function $\mathbf{r}(t)$ of the scalar independent variable t. Equation (10.1) is known as the law of motion of a particle. Thus, the law of motion (10.1) specifies the position of the particle at any moment in time.

The velocity vector $\mathbf{v} = \{v_x(t),\ v_y(t),\ v_z(t)\}$ and the acceleration vector $\mathbf{a} = \{a_x(t),\ a_y(t),\ a_z(t)\}$ are defined in terms of the appropriate derivatives thus:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}, \qquad (10.2)$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\,\mathbf{i} + \frac{d^2y}{dt^2}\,\mathbf{j} + \frac{d^2z}{dt^2}\,\mathbf{k}. \qquad (10.3)$$

The law of motion (10.1) is fundamental to kinematics. Knowing the law of motion, one can determine other physical quantities characterizing the motion of a particle, say, the components of the velocity vector $\mathbf{v}$ and the acceleration vector $\mathbf{a}$:

$$v_x(t) = \frac{dx}{dt}, \quad v_y(t) = \frac{dy}{dt}, \quad v_z(t) = \frac{dz}{dt}; \qquad (10.4)$$

$$a_x(t) = \frac{d^2x}{dt^2}, \quad a_y(t) = \frac{d^2y}{dt^2}, \quad a_z(t) = \frac{d^2z}{dt^2}. \qquad (10.5)$$

Hence, the law of motion (10.1) is directly related to the *basic problem of kinematics*. Formally there are two such problems: the *direct* and the *inverse*.

> The direct problem of kinematics consists in finding any parameter of motion from the known law of motion.

It is solved by consistently applying the basic laws of kinematics (10.1)-(10.3).

> The inverse problem of kinematics consists in determining the law of motion from a known parameter of motion (the velocity vector $\mathbf{v}$ or the acceleration vector $\mathbf{a}$).

The inverse problem is considerably more complicated than the direct problem. It can be shown that the great variety of kinematic problems can be reduced to these two types. Below we consider several examples of the direct and inverse problems of kinematics.

Example 10.1. *Determine the absolute value of the velocity of a particle at time $t = 2$ s if the particle moves according to the law* $\mathbf{r} = \alpha t^2\mathbf{i} + \beta (\sin \pi t)\mathbf{j}$, *with* $\alpha = 2$ m/s² *and* $\beta = 3$ m.

*Solution. A physical analysis.** The physical system consists of a single idealized object, a particle. The law of motion is formally specified. Hence, our problem belongs to the class of direct problems of kinematics (given the law of motion. find one of the parameters of motion, in our case the absolute value of the velocity vector). Using the known law of motion, we find that the components of the radius vector $\mathbf{r}(t)$ are

$$x(t) = \alpha t^2, \tag{10.6}$$

$$y(t) = \beta \sin \pi t, \tag{10.7}$$

$$z(t) = 0. \tag{10.8}$$

Thus, the particle moves in the XOY plane. Consequently, each of the vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ has two components. Combining the definition of the velocity vector with Eqs. (10.2), (10.4), (10.6), and (10.7), we find the following expressions for the velocity components:

$$v_x = 2\alpha t, \quad v_y = \beta\pi \cos \pi t.$$

This yields the following expression for the absolute value of the velocity vector:

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{4\alpha^2 t^2 + \beta^2\pi^2 \cos^2 \pi t}.$$

Susbstituting the necessary numerical values, we get $v \approx 12.4$ m/s.

* We will not carry out the introductory part of the physical analysis in full. Hence, the words "a physical analysis" following the word "solution" mean that only the main part of the method of analyzing the physical content of a problem is carried out (choice and analysis of the physical system, the study of the physical phenomenon, etc.).

Example 10.2. *A particle moves according to the law* $\mathbf{r} = \alpha (\sin 5t)\ \mathbf{i} + \beta (\cos^2 5t)\ \mathbf{j}$, *with* $\alpha = 2$ m *and* $\beta = 3$ m. *Find the velocity vector, acceleration vector, and trajectory of the particle's motion.*

Solution. This is also a direct problem of kinematics. We start with the components of the radius vector:

$$x(t) = \alpha \sin 5t, \tag{10.9}$$

$$y(t) = \beta \cos^2 5t, \tag{10.10}$$

$$z(t) = 0. \tag{10.11}$$

Thus, the particle moves in the XOY plane. Next we find the components of the velocity vector:

$$v_x(t) = 5\alpha \cos 5t, \tag{10.12}$$

$$v\ (t) = -5\beta \sin 10t. \tag{10.13}$$

Equations (10.12) and (10.13) yield the expressions for the components of the acceleration vector:

$$a_x(t) = -25\,\alpha \sin 5t, \tag{10.14}$$

$$a_y(t) = -50\,\beta \cos 10t. \tag{10.15}$$

To find the trajectory we exclude t from the system of equations (10.9)-(10.10). The result is

$$y = 3 - (3/4)\,x^2. \tag{10.16}$$

Hence, the particle moves along a parabola.

Example 10.3. *The velocity of a particle varies according to the law* $\mathbf{v} = \alpha (2t^3 - \beta)\ \mathbf{i} - \gamma (\sin 2\pi t/3)\ \mathbf{j}$, *with* $\alpha = 1$ m/s^4, $\beta = 1$ s^3, and $\gamma = 1$ m/s. *Find the law of motion if at the initial moment* $t = 0$ *the particle was at the origin* $\mathbf{r}_0 = \{0, 0, 0\}$.

*Solution. A physical analysis.** The physical system consists only of the particle. Formally the law of variation of the particle's velocity and the particle's initial position are specified. We are looking for the law of motion of the particle. Hence, the given problem constitutes an

* In what follows we drop the words "a physical analysis" after the word "solution."

inverse problem of kinematics (given a single parameter of motion, the velocity, find the law of motion). The law of motion $\mathbf{r} = \mathbf{r}(t)$ and the velocity vector $\mathbf{v}$ are related through the vector differential equation (10.2), which is equivalent to three differential equations (10.4). In our case the velocity components $v_x(t)$, $v_y(t)$, and $v_z(t)$ are known functions of time:

$$v_x = \alpha(2t^3 - \beta), \quad v_y = -\gamma(\sin 2\pi t/3), \quad v_z = 0.$$

Substituting these expressions for v_x, v_y, and v_z into Eqs. (10.4), we obtain a system of three differential equations in three unknown functions $x(t)$, $y(t)$, and $z(t)$, the components of the radius vector $\mathbf{r}$:

$$\alpha(2t^3 - \beta) = \frac{dx}{dt}, \quad -\gamma \sin\frac{2\pi}{3}t = \frac{dy}{dt}, \quad 0 = \frac{dz}{dt}.$$

Separating the variables and integrating, we find that

$$x = \alpha\left(\frac{1}{2}t^4 - \beta t\right) + c_1, \tag{10.17}$$

$$y = \frac{3\gamma}{2\pi}\cos\frac{2\pi}{3}t + c_2, \tag{10.18}$$

$$z = c_3, \tag{10.19}$$

where c_1, c_2, and c_3 are arbitrary constants that can be determined from the initial conditions. Allowing for the fact that $x = y = z = 0$ at $t = 0$ and employing the system of equations (10.17)-(10.19), we find that $c_1 = 0$, $c_2 = -3\gamma/2\pi$, and $c_3 = 0$. We can now write the final expressions for the components of the radius vector $\mathbf{r}$:

$$x(t) = \alpha\left(\frac{1}{2}t^4 - \beta t\right), \quad y = \frac{3\gamma}{2\pi}\left(\cos\frac{2\pi}{3}t - 1\right), \quad z = 0.$$

Thus, the law of motion has the form

$$\mathbf{r}(t) = \alpha\left(\frac{1}{2}t^4 - \beta t\right)\mathbf{i} + \frac{3\gamma}{2\pi}\left(\cos\frac{2\pi}{3}t - 1\right)\mathbf{j}.$$

Note that if we now solve the direct problem (given the law of motion, find the velocity), we can obtain the starting expression for the velocity vector:

$$\mathbf{v}(t) = \alpha(2t^3 - \beta)\mathbf{i} - \gamma(\sin 2\pi t/3)\mathbf{j}.$$

Knowing the law of motion we can find any parameter of motion: velocity $\mathbf{v}$, acceleration $\mathbf{a}$, the trajectory, etc.

Example 10.4. *The acceleration of a particle varies according to the law* $\mathbf{a} = \alpha t^2 \mathbf{i} - \beta \mathbf{j}$, *with* $\alpha = 3$ m/s^4 *and* $\beta = 3$ m/s^2. *Find the distance from the origin to the point where the particle will be at time* $t = 1$ s *if* $\mathbf{v}_0 = 0$ *and* $\mathbf{r}_0 = 0$ *at* $t = 0$.

Solution. The hypothesis of the problem shows that the particle moves in the XOY plane. To find the distance from the origin to the point where the particle will be at $t = 1$ s, we must know the particle's law of motion. Thus, we have an inverse problem of kinematics: given a parameter of motion (in our case this is acceleration $\mathbf{a}$), determine the law of motion $\mathbf{r} = \mathbf{r}(t)$ and then find the absolute value of $\mathbf{r}$ at $t = 1$ s.

We start by determining the velocity vector from Eq. (10.3):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}, \quad \text{or} \quad \mathbf{a} = \frac{dv_x}{dt}\,\mathbf{i} + \frac{dv_y}{dt}\,\mathbf{j}.$$

This vector differential equation is equivalent to the following two differential equations:

$$\frac{dv_x}{dt} = \alpha t^2, \quad \frac{dv_y}{dt} = -\beta.$$

Separating the variables and integrating, we arrive at the following expressions for the components of the velocity vector:

$$v_x = \frac{\alpha t^3}{3} + c_1, \quad v_y = -\beta t + c_2.$$

Allowing for the initial conditions ($v_x = v_y = 0$ at $t = 0$), we can find the values of the arbitrary constants: $c_1 = 0$ and $c_2 = 0$.

Next we turn to the system of differential equations

$$\frac{dx}{dt} = \frac{\alpha t^3}{3}, \quad \frac{dy}{dt} = -\beta t$$

and find the components $x(t)$ and $y(t)$ of the radius vector $\mathbf{r}(t)$:

$$x(t) = \frac{\alpha t^4}{12} + c_3, \quad y(t) = -\frac{\beta t^2}{2} + c_4, \qquad (10.20)$$

where c_3 and c_4 are arbitrary constants. By allowing for the initial conditions ($x = y = 0$ at $t = 0$) we find from Eqs. (10.20) that $c_3 = c_4 = 0$. We have now found the law of motion:

$$\mathbf{r}(t) = \frac{\alpha t^4}{12}\,\mathbf{i} - \frac{\beta t^2}{2}\,\mathbf{j}. \tag{10.21}$$

If we use the formula for the absolute value of a vector, or the length of a vector, we can determine the sought distance from the origin to the particle at $t = 1$ s:

$$|\mathbf{r}| = \sqrt{x^2 + y^2},$$

which yields $r \approx 1.52$ m.

Analysis of the solution. Knowing the law of motion, we can find any parameter that characterizes the motion of a particle and, hence, we can formulate and solve many other kinematic problems concerned with finding these parameters. Let us formulate, for example, the problem of finding the trajectory of a given particle: *given the acceleration* $\mathbf{a} = \alpha t^2\mathbf{i} - \beta\mathbf{j}$ *and the same initial conditions* (these could be different as well), *find the trajectory described by the particle.* After the law of motion (10.21) has been obtained, the trajectory of the particle can be found by excluding time t from the system of equations

$$x = \frac{\alpha t^4}{12}, \quad y = -\frac{\beta t^2}{2}.$$

The set of methods for solving the direct and inverse problems of kinematics constitutes the essence of the kinematic method mentioned in Section 6.

Very often an arbitrary kinematic problem with real content can be reduced to the schematized direct and inverse problems of kinematics just discussed. Let us demonstrate this assertion using a concrete example.

Example 10.5. *A train is moving rectilinearly with a velocity* $v_0 = 180$ km/hr. *Suddenly an obstacle appears on the tracks and the engineer brakes. Now the velocity of the train changes according to the law* $v = v_0 - \alpha t^2$, *with* $\alpha = 1$ m/s^3. *What is the braking distance of the train? How much time must pass from the moment the brakes are applied before the train stops?*

Solution. The physical system consists of only one object, the train, and the train can be thought of as a particle. The motion of the train is studied formally, without clarifying what caused the change in its state of motion (the principles on which the brakes operate are assumed to be unknown, and knowledge of these principles is unimportant to the solution of the problem). We know the law of variation of one of the parameters of motion, the velocity. We must find other physical quantities characterizing the train's motion (the braking distance and time). Thus, we have an inverse problem of kinematics, which can be formulated in the following manner: *given the velocity of a particle as a function of time,* $\mathbf{v} = (v_0 - \alpha t^2)\,\mathbf{i}$, *with* $\alpha = 1$ m/s^3 *and* $v_0 = 180$ km/hr, *find the time it takes the particle to stop and the distance the particle travels before it stops if at* $t = 0$ *we have* $\mathbf{r} = 0$ *and* $\mathbf{v}_0 = v_0\mathbf{i}$ (the latter condition follows from the law of variation of velocity $\mathbf{v} = (v_0 - \alpha t^2)\,\mathbf{i}$). When the problem is formulated in this manner, it is unimportant what real object is moving: a train, car, motorboat, or submarine (one only has to change the constant parameters α and v_0).

We solve this inverse problem of kinematics by employing the kinematic method. Since the motion of the particle is one-dimensional (along the X axis), to find the law of motion we need only one differential equation:

$$dx = v\,dt, \text{ or } dx = (v_0 - \alpha t^2)\,dt.$$

After integrating the latter equation and allowing for the initial conditions, we arrive at the law of motion,

$$x = v_0 t - \alpha t^3/3. \tag{10.22}$$

The time it takes the train to come to a stop can be calculated by nullifying the train's velocity:

$$0 = v_0 - \alpha t^2.$$

This yields $t = \sqrt{v_0/\alpha}$. Substitution of the necessary numerical values yields $t \approx 7$ s. From Eq. (10.22) we find the braking distance: $x \approx 230$ m.

Example 10.6. *A rocket is launched from a land-based site vertically with an acceleration* $a = \alpha t^2$, *with* $\alpha = 1$ m/s^4.

At an altitude $h_0 = 100$ km above Earth's surface the rocket's boosters fail. How much time will pass (from the moment the boosters failed) before the rocket crashes? Air drag is ignored. The initial velocity of the rocket is $v_0 = 0$.

Solution. The rocket can be thought of as a particle. We know the initial conditions and how the acceleration varies with time. We must find other physical characteristics of the rocket's motion (velocity, time of motion, and the law of motion). This constitutes an inverse problem of kinematics. After integrating the equation $dv = a\,dt$ and allowing for the initial conditions, we find the law of time variation of the velocity:

$$v = \alpha t^3/3.$$

After integrating a second time and allowing once more for the initial conditions, we arrive at the law of motion of the rocket:

$$h = \alpha t^4/12.$$

These laws are valid only up to the moment when the boosters failed. Let us determine the velocity of the rocket at that moment (this velocity constitutes the initial velocity in the rocket's further motion):

$$v_{01} = \frac{\alpha}{3} \sqrt[4]{\left(\frac{12h_0}{\alpha}\right)^3}.$$

This yields $v_{01} \approx 12.1$ km/s, which exceeds the escape velocity of objects launched from Earth, roughly 11.2 km/s. Hence, the rocket will never return to Earth.

11. Particle Dynamics

When we study the dynamics of the motion of objects, we must introduce a new concept that allows for interactions between the objects. This is the concept of *force*, F. But how do we find the forces acting on an object?

Let us first establish with what other objects a given object interacts. Then we can find how the given object interacts with the others and in what ways (type of interaction).

As noted earlier, for classical physical systems the following types of interaction play an important role: the gravitational (Newton's law of gravitation $F = Gm_1m_2/r^2$) and the electromagnetic (its particular manifestations are the Coulomb force $F = Q_1Q_2/4\pi\varepsilon_0\varepsilon r^2$, the Lorentz force $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$, the friction force $F_{fr} = fN$, and the elastic force $F = -kx$). Thus, it is only as the result of the interaction of a given object with another object that several different forces may act on the given object. It is important to understand that these forces differ qualitatively. The next step is to evaluate each force quantitatively, that is, determine the order of magnitude of each force. It may so happen that some forces are so weak that they can be ignored in the conditions of a specific problem.

Example 11.1. *Two objects of masses* $m_1 = 1$ kg *and* $m_2 = 2$ kg *are connected by a massless string and move in the horizontal plane (on Earth) under a force* $F = 10$ N

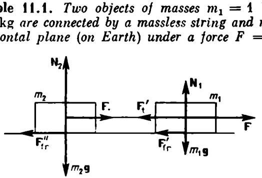

Figure 11.1

directed horizontally and applied to object m_1 *(Figure 11.1). Determine the forces acting on each object if the coefficient of friction between* m_1 *and* m_2 *and the horizontal surface is* $f = 0.5$.

Solution. Consider the object m_1. Force **F** is applied to it, as the hypothesis states. What are the other forces acting on m_1? Object m_1 interacts with Earth, the string, and object m_2. With Earth m_1 interacts through Newton's law of gravitation and, hence, the force of gravity $m_1\mathbf{g}$ acts on object m_1 downward. Next, object m_1 inter-

acts with Earth elastically, and this fact manifests itself in the appearance of an elastic force $\mathbf{N}_1$ exerted by the support (the horizontal plane) on object m_1. This force points upward. Furthermore, as a result of the interaction of object m_1 with Earth there appears a force of friction $F'_{fr} = fN_1$. Finally, object m_1 can interact with the string only elastically, that is, there is a tensile force $\mathbf{F}'_t$ pointing to the left and acting on object m_1 (since the string is massless, the gravitational interaction between m_1 and string is nil). Object m_1 can interact with object m_2 only via Newton's law of gravitation, but this interaction is so weak that in the conditions of the present problem it can be ignored. Thus, there are five forces acting on object m_1: $\mathbf{F}$, $m_1\mathbf{g}$, $\mathbf{N}_1$, $\mathbf{F}'_{fr}$, and $\mathbf{F}'_t$.

Reasoning in the same manner, we can demonstrate that there are four forces acting on object m_2: the tensile force $\mathbf{F}'_t$, the force of gravity $m_2\mathbf{g}$, the elastic force $\mathbf{N}_2$ exerted by the support (the horizontal plane), and the force of friction $F''_{fr} = fN_2$. Only two elastic tensile forces $\mathbf{F}'_t$ and $\mathbf{F}''_t$ act in the massless and nonexpandable string. On the basis of Newton's second law,

$$\mathbf{F} = m\mathbf{a}, \tag{11.1}$$

we conclude that these forces are equal in magnitude, $F'_t = F''_t$ (since the string is massless, or $m = 0$, according to Newton's second law we have for the string $F'_t - F''_t = 0 \times a$, or $F'_t = F''_t$).

Newton's second law is the fundamental law of particle dynamics. It is valid only in an inertial reference frame for a single object (a particle).

In the particular case where objects move with speeds considerably lower than the speed of light in empty space ($v \ll c$), Newton's second law can be rewritten as

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}, \tag{11.2}$$

or

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}. \tag{11.3}$$

If Newton's second law is written for a noninertial reference frame, the right-hand sides of Eqs. (11.1)-(11.3) contain forces of inertia.

The content (or physical meaning) of the fundamental laws (11.1)-(11.3) lies in the fact that a change in momentum $m\mathbf{v}$ or velocity $\mathbf{v}$ of a particle is due to and determined by the action of forces. Hence, if we know the forces and the initial conditions (the position and velocity of a particle at the initial moment of time), we can find the variations in the particle's motion. This constitutes the basic (ideal) *problem of dynamics*: given the forces and initial conditions, determine the change in the motion of the system (the mechanical state of the system).

To find the change in the motion of an object, we must know the law of motion of the object. Determining the law of motion from a known parameter of the motion (and the initial conditions) constitutes, as we have just established, the essence of an inverse problem of kinematics. In dynamics a parameter of the motion of a particle can be found by consistently applying Newton's second law to describe the motion of each object in the system. This law is written in the form

$$\mathbf{a} = \mathbf{F}/m \tag{11.4}$$

(then the acceleration vector $\mathbf{a}$ of each object is determined and, by solving the inverse problem of kinematics, the law of motion is established) or in the form (11.2) (then the velocity vector of each object is determined and, by solving the inverse problem of kinematics, the law of motion is established) or in the form (11.3) (then the law of motion is established directly by solving this differential equation).

To write Newton's second law correctly in each specific case, we must know the method of application of this law. This method has been discussed with sufficient detail in Section 1.

Example 11.2. *A massless pulley is fastened to the top of a wedge whose mass is* $m_3 = 10$ kg (*Figure 11.2*). *A massless and nonexpandable string is flung over the pulley, and the ends of the string are tied to blocks of masses* $m_1 = 1$ kg *and* $m_2 = 10$ kg. *The coefficients of friction of blocks* m_1 *and* m_2 *against the faces of the wedge are* $f_1 = 0.2$ *and* $f_2 = 0.1$, *respectively, and the coefficient of friction of the wedge against the horizontal plane is* $f_3 = 0.3$.

The angles formed by the faces of the wedge with the horizontal surface are $\alpha_1 = 30°$ and $\alpha_2 = 60°$, respectively. Find the tensile stress developed by the string.

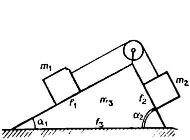

Figure 11.2

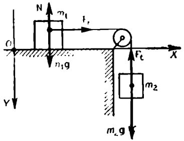

Figure 11.3

Solution. The problem is complicated. Let us try to simplify it and then lift the simplifying assumptions one by one.

Let us assume that (a) $f_1 = 0$, (b) $\alpha_1 = 0$, (c) $\alpha_2 = 90°$, and (d) $f_3 = \infty$ (the wedge is fixed to the horizontal surface). Then we get a relatively simple problem which can be formulated as follows: *two objects (blocks) of mass* $m_1 = 1$ kg *and* $m_2 = 10$ kg *are fastened to the ends of a massless, nonexpandable string flung over a massless pulley (Figure 11.3); object m_1 can move along a smooth, fixed horizontal surface; find the tensile stress developed by the string.*

Let us solve this simplified problem. The physical system will incorporate four objects: block m_1, block m_2, the string, and the pulley. Blocks m_1 and m_2 may be thought of as particles. The mechanical motion of the objects constitutes the physical phenomenon occurring in the system. The objects constituting the system interact between themselves and with external objects (the horizontal plane and Earth). Owing to these forces the objects of the system (except the pulley) move rectilinearly and with uniform acceleration. Thus, we have a basic problem of dynamics. For its solution we employ Newton's second law in the form (11.4). This law can

be applied only to particles m_1 and m_2 (the string and the pulley are not particles). Let us take the support as the inertial reference frame and send the X and Y axes as shown in Figure 11.3.

Consider particle m_1. The following forces act on it: the force of gravity $m_1\mathbf{g}$ (which emerges as the result of the interaction of mass m_1 with Earth through Newton's law of gravitation), force $\mathbf{N}$ exerted by the support (the elastic force of interaction of m_1 with the support, which is the horizontal plane), and the tensile stress $\mathbf{F}_t$ developed by the string (due to the elastic interaction of particle m_1 and the string). The other forces are weak. The above-mentioned forces have already been projected onto the X and Y axes. Hence, we can immediately write Newton's second law in the form of two equations in terms of the projections on the X and Y axes:

$$m_1 a_{1x} = F_t, \tag{11.5}$$

$$m_1 a_{1y} = m_1 g - N, \tag{11.6}$$

where a_{1x} and a_{1y} are the projections of the acceleration vector $\mathbf{a}_1$ of particle m_1 on the X and Y axes. Since $a_{1y} = 0$, we have $N = m_1 g$.

Let us now consider particle m_2. Two forces act on this particle, the force of gravity $m_2\mathbf{g}$ and the tensile stress $\mathbf{F}_t$ developed by the string. Figure 11.3 shows that the projection of these forces on the X axis is zero and that the algebraic sum of the projections of these forces on the Y axis is $m_2 g - F_t$. Hence, by Newton's second law for m_2 we obtain

$$m_2 a_{2y} = m_2 g - F_t, \tag{11.7}$$

where a_{2y} is the projection of the acceleration vector $\mathbf{a}_2$ of m_2 on the Y axis. It can easily be demonstrated that the projection of $\mathbf{a}_2$ on the X axis is nil ($a_{2x} = 0$). Since $a_{1x} = a_{2y} = a$, the system of equations (11.5)-(11.7) acquires the form

$$m_1 a = F_t, \tag{11.8}$$

$$m_2 a = m_2 g - F_t. \tag{11.9}$$

Thus, we have set up a closed system of two equations with two unknowns (a and F_t). Physically the problem

has been solved, that is, the physical stage of the solution has been completed.

Solving the derived system of equations (11.8), (11.9), we arrive at an answer in general form:

$$a = \frac{m_2}{m_1 + m_2} g, \tag{11.10}$$

$$F_t = \frac{m_1 m_2}{m_1 + m_2} g. \tag{11.11}$$

Substituting the necessary numerical values, we arrive at $a \approx 8.9$ m/s^2 and $F_t \approx 8.9$ N. We have thus completed the mathematical stage of the solution.

It is now advisable to go through the last stage of the solution, the analysis. Formula (11.10) shows that the acceleration of the system depends on both the value of m_1 and the value of m_2. Let us consider two limiting cases: (1) $m_1 \gg m_2$, and (2) $m_1 \ll m_2$. In the first $a \approx g m_2/m_1$, that is, acceleration a is low (a small object m_2 pulls an extremely large object m_1). In the second case $a \approx g$, that is, the system moves thanks to the large object m_2 with almost the maximally possible (in the given case) acceleration equal to g. In the same manner we can analyze, via (11.11), the dependence of the tensile stress F_t on the values of m_1 and m_2.

Now let us lift the simplifying assumptions. (a) Suppose that the friction coefficient f_1 is not zero. Then there appears an additional force acting on object m_1, the friction force $F_{fr} = f_1 N_1$ pointing in the negative direction of the X axis. The conditions in which object m_2 operates remain unchanged. Applying Newton's second law to each object, we arrive at a closed system of equations:

$$m_1 a = F_t - f_1 m_1 g, \quad m_2 a = m_2 g - F_t.$$

Solving the system, we get

$$a = \frac{m_2 - f_1 m_1}{m_1 + m_2} g, \tag{11.12}$$

$$F_t = \frac{m_1 m_2 (1 + f_1)}{m_1 + m_2} g, \tag{11.13}$$

which yield $a \approx 8.74$ m/s^2 and $F_t \approx 10.68$ N.

If we compare (11.12) with (11.10) and (11.13) with (11.11), we see that allowing for friction forces diminishes the acceleration of the system (by what factor and on what does this reduction depend?) and increases the tensile stress developed by the string.

(b) Suppose that $\alpha_1 \neq 0$ and $f_1 \neq 0$. The conditions in which block m_2 operates remain unchanged. The forces acting on m_1 and m_2 are shown in Figure 11.4.

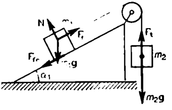

Figure 11.4

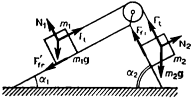

Figure 11.5

Allowing for the fact that $N_1 = m_1 g \cos \alpha_1$ and $F_{fr} = f_1 N_1 = f_1 m_1 g \cos \alpha_1$, we arrive at the following closed system of equations:

$$m_1 a = F_t - f_1 m_1 g \cos \alpha_1 - m_1 g \sin \alpha_1,$$

$$m_2 a = m_2 g - F_t.$$

Solving the system, we get

$$a = \frac{m_2 - f_1 m_1 \cos \alpha_1 - m_1 \sin \alpha_1}{m_1 + m_2} g,$$

$$F_t = \frac{m_1 m_2 (1 + f_1 \cos \alpha_1 + \sin \alpha_1)}{m_1 + m_2} g,$$

which yield $a \approx 8.32$ m/s^2 and $F_t \approx 14.9$ N. Thus, the acceleration has further diminished and the tensile stress has increased.

(c) Suppose now that $\alpha_1 \neq 0$, $\alpha_2 \neq 90°$, $f_1 \neq 0$, and $f_2 \neq 0$. The forces acting on objects m_1 and m_2 are depicted in Figure 11.5. Newton's second law as applied to objects m_1 and m_2 yields

$$m_1 a = F_t - F'_{fr} - m_1 g \sin \alpha_1, \tag{11.14}$$

$$m_2 a = m_2 g \sin \alpha_2 - F_{r} - F_t \tag{11.15}$$

Allowing for the fact that $N_1 = m_1 g \cos \alpha_1$, $N_2 = m_2 g \cos \alpha_2$, $F'_{fr} = f_1 N_1 = f_1 m_1 g \cos \alpha_1$, and $F''_{fr} = f_2 N_2 = f_2 m_2 g \cos \alpha_2$, we can write the system of equations (11.14), (11.15) as follows:

$$m_1 a = F_t - f_1 m_1 g \cos \alpha_1 - m_1 g \sin \alpha_1,$$

$$m_2 a = m_2 g \sin \alpha_2 - f_2 m_2 g \cos \alpha_2 - F_t.$$

Solving this system, we get

$$a = \frac{m_2 \sin \alpha_2 - f_1 m_1 \cos \alpha_1 - f_2 m_2 \cos \alpha_2 - m_1 \sin \alpha_1}{m_1 + m_2} g, \quad (11.16)$$

$$F_t = \frac{m_1 m_2 (\sin \alpha_1 + f_1 \cos \alpha_1 + \sin \alpha_2 - f_2 \cos \alpha_2)}{m_1 + m_2} g, \quad (11.17)$$

which yield $a \approx 6.62$ m/s^2 and $F_t \approx 13.2$ N.

We see from (11.16) and (11.17) that acceleration a has further diminished and the tensile stress developed by the string has also diminished in comparison with its value in case (b). The case (d) (the wedge is not fixed to the horizontal surface) will be considered later.

Equations (11.16) and (11.17) show that the sought quantities (acceleration a and tensile stress F_t) depend in a very complicated manner on the other parameters of the physical system: the masses m_1 and m_2, the angles α_1 and α_2, and the coefficients of friction f_1 and f_2. This dependence can be studied analytically.

Knowing one of the kinematic quantities of a physical system (say, acceleration a), we can arrive at the law of motion by solving the inverse problem of kinematics. If the initial velocity of the system is zero, the law has the form $x = x_0 + at^2/2$, where x_0 is the initial position of one of the two blocks. Hence, one can find the velocity of any block in the system at an arbitrary time t, the position of each block in space, and other physical quantities characterizing the objects in the system and the physical phenomena occurring in it. For instance, we can find the momenta of blocks m_1 and m_2 in the system ($p_1 = m_1 v_1$, $p_2 = m_2 v_2$), the values of the respective kinetic energies $E_{k1} = m_1 v_1^2/2$ and $E_{k2} = m_2 v_2^2/2$, and so on.

Thus, by solving the basic problem of dynamics (this amounts to finding the law of variation of one kinematic quantity (acceleration $\mathbf{a}(t)$, velocity $\mathbf{v}(t)$, or the radius vector $\mathbf{r}(t)$), we can determine the mechanical state of a physical system.

The problem we have just solved can be made still more complicated by assuming, say, that not two but three or more blocks are connected by strings, that there are two or more pulleys, that the wedge has three or more faces (instead of two) along which the blocks can move, and so on. In short, we can formulate dozens of similar problems whose principal idea is the same. It is important to note that all these can be solved by the same dynamical method. In Example 11.2 we considered several problems of varying complexity, but in their essence they were all the same problem and were solved by applying a single dynamical method.

Note that the problems solved in Example 11.2 had one very characteristic feature in common, that is, the forces acting on the objects in the system were all constant. In all problems of this kind the law of motion can be formulated in advance: if the motion occurs along the X axis, the law is $x = x_0 + v_{0x}t + a_x t^2/2$ (similar equations can be written for the movements along other axes). Thus, the motion of objects in this case is uniformly accelerated (or uniformly decelerated).

Let un now consider examples of more complicated problems involving forces that are not constant.

Example 11.3. *A skydiver of mass $m = 100$ kg is making a delayed drop with an initial speed $v_0 = 0$. Find the law by which the skydiver's speed varies before the parachute is opened if the air drag is proportional to the skydiver's speed,* $\mathbf{F}_d = -k\mathbf{v}$, *with* $k = 20$ kg/s.

Solution. The physical system in the given case consists of a single object, the skydiver, which can be considered as being a particle. The physical phenomenon studied here is the motion of the particle as a result of the particle's interaction with external objects (Earth and air). We must find one of the kinematic parameters of the particle's motion, the speed as a function of time.

This is a basic problem of dynamics. We apply Newton's second law (the conditions of applicability of this law are met). For the inertial reference frame we take Earth (Figure 11.6). We place the origin at the point where the skydiver begins the jump, point O, and send the X axis downward. Since the altitude h is small compared to Earth's radius, the acceleration of free fall may be assumed constant: $g \approx 9.8\ \mathrm{m/s^2} = \mathrm{const}$. Two forces act on the skydiver, the force of gravity $m\mathbf{g}$ and the air drag $\mathbf{F}_d = -k\mathbf{v}$. Newton's second law enables us to obtain a differential equation for the unknown function $v(t)$:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv.$$

Separating the variables, we obtain

$$-\frac{\mathrm{d}v}{mg/k - v} = -\frac{k}{m}\,\mathrm{d}t, \quad \text{or} \quad \frac{\mathrm{d}\,(mg/k - v)}{mg/k - v} = -\frac{k}{m}\,\mathrm{d}t.$$

Integration yields

$$\ln(mg/k - v) = -(k/m)\,t + c. \qquad (11.18)$$

We find the arbitrary constant c by employing the initial conditions ($v = v_0 = 0$ at $t = 0$), which yield $c = \ln(mg/k)$. Substituting this value of constant c into Eq. (11.18) and performing relatively simple manipulations, we find the law of variation of the skydiver's speed of fall:

$$v = \frac{mg}{k}\left(1 - \mathrm{e}^{-(k/m)t}\right). \qquad (11.19)$$

Equation (11.19) demonstrates that as t tends to infinity the speed tends to its maximal value $v_{\max} = mg/k$, which amounts to about 50 m/s. Experience has shown that it takes the skydiver a relatively short time interval to achieve this speed and after that the skydiver approaches Earth's surface uniformly at this speed. Theoretically, the fall of a skydiver is always accelerated (the speed grows continuously), but starting at a certain moment in time the change in the skydiver's speed can be ignored and the skydiver can be assumed to be falling uniformly (at a constant speed).

Since the law of variation of the skydiver's speed is known, we can find the law of motion of the skydiver by solving an inverse problem of kinematics:

$$\mathrm{d}x = v(t)\,\mathrm{d}t, \quad x(t) = \int v(t)\,\mathrm{d}t,$$
$$x = \frac{mg}{k}t - \frac{m^2 g}{k^2}\left(1 - \mathrm{e}^{-(k/m)t}\right). \tag{11.20}$$

In finding the law of motion (11.20) we employed the initial condition, $x = 0$ at $t = 0$, to determine the arbitrary constant.

Thus, the solution of the problem is complete.

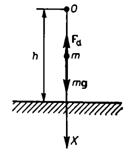

Figure 11.6

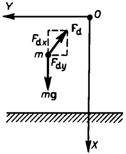

Figure 11.7

Now let us make the initial conditions of the problem more complicated: suppose that $x = y = 0$ at $t = 0$ but the initial velocity has a horizontal component, $\mathbf{v}_0 = \{0, v_0\}$. In this case the skydiver's path of fall is curvilinear (Figure 11.7). As formerly, two forces act on the skydiver: the force of gravity $m\mathbf{g}$ and the air drag $\mathbf{F}_d = -k\mathbf{v}$. But now the air drag $\mathbf{F}_d$ is directed along the tangent to the path and, hence, we must allow for the vector nature of Newton's second law. Projecting the air drag $\mathbf{F}_d$ on the X and Y axes and employing Newton's second law, we obtain

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = mg - kv_x, \tag{11.21}$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -kv_y, \tag{11.22}$$

where v_x and v_y are the unknown components of the velocity vector **v**.

Separating the variables in Eqs. (11.21) and (11.22) and integrating with allowance made for the initial condition ($v_x = 0$ and $v_y = v_0$ at $t = 0$), we get

$$v_x = \frac{mg}{k}\left(1 - e^{-(k/m)t}\right), \tag{11.23}$$

$$v_y = v_0 e^{-(k/m)t}. \tag{11.24}$$

Let us find the law of motion of the skydiver. Substituting the v_x and v_y values from (11.23) and (11.24) into the relationships $dx = v_x dt$ and $dy = v_y dt$, we obtain two differential equations for determining the two unknown functions, the $x(t)$ and $y(t)$ components of the radius vector $\mathbf{r}(t)$:

$$dx = \frac{mg}{k}\left(1 - e^{-(k/m)t}\right) dt, \quad dy = v_0 e^{-(k/m)t}\, dt.$$

After integrating these equations and allowing for the initial condition ($x = y = 0$ at $t = 0$), we arrive at the law of motion of the skydiver in parametric form in the form of two equations:

$$x = \frac{mg}{k}\, t - \frac{m^2 g}{k^2}\left(1 - e^{-(k/m)t}\right), \tag{11.25}$$

$$y = \frac{mv_0}{k}\left(1 - e^{-(k/m)t}\right). \tag{11.26}$$

The law of motion can, of course, also be written in vector form:

$$\mathbf{r}(t) = \left[\frac{mg}{k}\, t - \frac{m^2 g}{k^2}\left(1 - e^{-(k/m)t}\right)\right]\mathbf{i} + \left[\frac{mv_0}{k}\left(1 - e^{-(k/m)t}\right)\right]\mathbf{j}.$$

Now, knowing the law of motion, we can determine any parameter characterizing the given mechanical phenomenon; for one thing, excluding time t from the system of equations (11.25), (11.26), we arrive at the equation describing the trajectory of the skydiver:

$$x = -\frac{m^2 g}{k^2}\left[\ln\left(1 - \frac{k}{mv_0}\, y\right) + \frac{k}{mv_0}\, y\right].$$

Thus, this complicated problem too has been fully solved.

The forces acting on a moving object may depend not only on the object's velocity but on time t, coordinates x, y, and z, etc. Let us consider such a problem.

Example 11.4. *The thrust of a braking engine is proportional to time, $F = -kt$, where $k = \text{const}$. Neglecting friction (and air drag), calculate the time it will take an object of mass m, with the braking engine mounted on the object, to come to a halt. The object's speed just before the engine was turned on was v_0. It is assumed that the mass of the engine is much smaller than that of the object.*

Solution. The physical system consists of a single object of mass m, which can be assumed to be a particle.

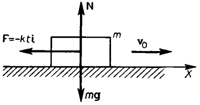

Figure 11.8

The physical phenomenon occurring in this system amounts to the object moving as a result of the interaction with other objects. We must find one of the parameters of the system, the duration of motion, t_1. The initial conditions are obvious: $v = v_0$ and $x = 0$ at $t = 0$. The trajectory of the object's motion is a straight line (i.e. the motion is one-dimensional). The final speed of the object is zero, or $v = 0$ at $t = t_1$. Thus, the problem considered here constitutes a basic problem of dynamics.

Let us use Newton's second law (its applicability conditions are met). We take Earth to be the inertial reference frame. There are three forces acting on the object (Figure 11.8): the force of gravity $m\mathbf{g}$, the elastic force $\mathbf{N}$ exerted by the support on the object (these two forces balance each other), and the thrust $\mathbf{F} = -kt\mathbf{i}$ of the braking engine (the nature of this force is immaterial to mechanics). Applying Newton's second law, we ar-

rive at a differential equation for one unknown function of time t, the speed $v(t)$:

$$m\frac{dv(t)}{dt} = -kt.$$

Separating the variables, integrating, and allowing for the initial conditions ($v = v_0$ at $t = 0$), we find the law of variation of the object's speed:

$$v = v_0 - kt^2/2m. \tag{11.27}$$

If in this equation we set the final speed v equal to zero, we arrive at an equation for finding the duration of motion, t_1:

$$0 = v_0 - kt_1^2/2m.$$

This leads us to the formula for t_1:

$$t_1 = \sqrt{2mv_0/k}. \tag{11.28}$$

After analyzing the solution, we can formulate other problems, say, finding the braking distance. To determine the braking distance x_1 we must know the law of motion. The law can be found by solving an inverse problem of kinematics:

$$x = v_0t - kt^3/6m. \tag{11.29}$$

Substituting into the law of motion (11.29) the value of the time t_1 taken from (11.28), we find the braking distance:

$$x_1 = \frac{2}{3}v_0\sqrt{\frac{2mv_0}{k}}.$$

Knowing the law of motion of an object, we can determine any parameters in a given physical phenomenon. The problem can easily be made more complicated by allowing, say, for friction.

We have thus considered several different problems in particle dynamics. In some the forces were constant, while in others the forces varied, but the approach to all was the same: the use of the three Newton's laws of dynamics (especially the second) to determine one parameter of motion (velocity **v**, acceleration **a**). The next

step in finding the law of motion usually involved solving an inverse problem of kinematics.

> The combined use of the three laws of Newton (especially the second law) constitutes the essence of the dynamical method of solving physics problems.

This method can be generalized so as to incorporate the case of noninertial reference frames. In Example 11.2 we did not consider case (d). Suppose that the wedge is not fixed to the support, i.e. $f_3 \neq \infty$. Now in general the wedge moves with an acceleration (a_3) with respect to the support (Earth) and no inertial reference frame can be associated with it. If all the conditions of this problem are taken into account, the problem becomes extremely complicated (not in principle but technically). Therefore, to illustrate the essence of applying the dynamical method when a noninertial reference frame is involved, let us simplify the problem as much as possible. We assume that friction is absent: $f_1 = f_2 = f_3 = 0$. Next, we assume that m_2, the string, and the pulley are also absent and that $\alpha_2 = 90^\circ$ (the last condition is unessential: angle α_2 may have any magnitude). Thus, we are considering the following problem:

Example 11.5. *A particle of mass* $m_1 = 1$ kg *is placed on the inclined surface of a smooth wedge of mass* $m_3 = 10$ kg. *The wedge can move along a smooth horizontal surface. The angle* α_1 *at the base of the wedge is* 30°. *Determine the accelerations of wedge and block.*

Solution. Two objects will constitute the physical system: the particle m_1 and the wedge m_3. A wedge cannot be thought of as a particle, but in the conditions of the present problem (the wedge moves rectilinearly) we can assume approximately that, first, all the forces applied to the wedge are applied at the wedge's center of mass and, second, that Newton's second law can be applied to the wedge.

The physical phenomenon occurring in this system is the motion of the two objects, m_1 and m_3, where particle m_1 moves with respect to the wedge, and wedge m_3 moves with respect to Earth. We must find the kine-

matic characteristics of this phenomenon, the acceleration of particle, $\mathbf{a}_1$, with respect to the wedge and the acceleration of wedge, $\mathbf{a}_3$, with respect to Earth. This constitutes a basic problem of dynamics.

Let us start by studying the motion of the wedge with respect to an inertial reference frame, Earth. The X and Y axes are directed as shown in Figure 11.9. Three forces

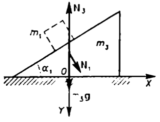

Figure 11.9

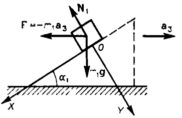

Figure 11.10

act on the wedge: the force of gravity $m_3\mathbf{g}$ (due to the interaction between wedge and Earth via Newton's law of gravitation), the force $\mathbf{N}_3$ exerted by the support on the wedge (due to the elastic interaction between wedge and Earth), and the force $\mathbf{N}_1$ (due to the elastic interaction of particle and wedge). By Newton's second law,

$$m_3\mathbf{a}_3 = m_3\mathbf{g} + \mathbf{N}_1 + \mathbf{N}_3. \tag{11.30}$$

Projecting Eq. (11.30) on the coordinate axes, we get

$$m_3a_{3x} = N_1 \sin \alpha_1,$$

$$m_3a_{3y} = m_3g + N_1 \cos \alpha_1 - N_3,$$

where a_{3x} and a_{3y} are the components of the acceleration vector $\mathbf{a}_3$ along the X and Y axes, respectively. Since $a_{3y} = 0$ and therefore $a_{3x} = a_3$, we get

$$m_3a_3 = N_1 \sin \alpha_1, \tag{11.31}$$

$$0 = m_3g + N_1 \cos \alpha_1 - N_3. \tag{11.32}$$

We have an open system of two equations in three unknowns: a_3, N_1, and N_3.

To find a closed system of equations let us study the motion of the particle with respect to the wedge. Since the wedge moves in an accelerated manner, the reference

frame linked with it is noninertial. We direct the coordinate axes as shown in Figure 11.10. Written in a noninertial reference frame, Newton's second law assumes the form

$$m\mathbf{a} = \sum \mathbf{F} + \mathbf{F}_i, \tag{11.33}$$

where the summation means the vector sum of "common" forces acting on an object of mass m moving with an acceleration $\mathbf{a}$ with respect to a noninertial reference frame, and $\mathbf{F}_i$ is the force of inertia, which in our case (translational motion) is equal to $-m_1\mathbf{a}_3$. Three forces act on the particle: the force of gravity $m_1\mathbf{g}$ (due to the interaction between particle and Earth via Newton's law of gravitation), the force $\mathbf{N}_1$ exerted by the wedge on the particle (due to the elastic interaction between particle and wedge), and the force of inertia $\mathbf{F}_i$. By Newton's second law (11.33),

$$m_1\mathbf{a}_1 = m_1\mathbf{g} + \mathbf{N}_1 + \mathbf{F}_i.$$

Projecting this equation on the coordinate axes, we get

$$m_1a_{1x} = m_1g\sin\alpha_1 + m_1a_3\cos\alpha_1, \tag{11.34}$$

$$m_1a_{1y} = m_1g\cos\alpha_1 + m_1a_3\sin\alpha_1 - N_1. \tag{11.35}$$

Since $a_{1y} = 0$ and, therefore, $a_{1x} = a_1$, from (11.34) and (11.35) we get

$$m_1a_1 = m_1g\sin\alpha_1 + m_1a_3\cos\alpha_1, \tag{11.36}$$

$$0 = m_1g\cos\alpha_1 - m_1a_3\sin\alpha_1 - N_1. \tag{11.37}$$

The system of four equations, (11.31), (11.32), (11.36), and (11.37), is closed (the unknowns are N_1, N_3, a_1, and a_3). Solving this system of equations, we find the sought quantities:

$$N_1 = \frac{m_1m_3g\cos\alpha_1}{m_3+m_1\sin^2\alpha_1}, \quad N_3 = m_3g\left(1+\frac{m_1\cos^2\alpha_1}{m_3+m_1\sin^2\alpha_1}\right),$$

$$a_1 = g\sin\alpha_1 + \frac{m_1g\cos^2\alpha_1}{m_1\sin\alpha_1 + m_3/\sin\alpha_1},$$

$$a_3 = \frac{m_1g\cos\alpha_1}{m_1\sin\alpha_1 + m_3/\sin\alpha_1},$$

which yield $N_1 \approx 8.2$ N, $N_3 \approx 105$ N, $a_1 \approx 5.25$ m/s², and $a_3 \approx 0.41$ m/s².

12. Mechanical Oscillations

The most common types of oscillations considered in the general physics course are *free continuous oscillations*, *free damped oscillations*, and *forced oscillations*. A characteristic feature of oscillatory motion is that such motion occurs under the action of variable forces. Consequently, after applying Newton's second law, we are left with a differential equation (usually not in vector form since one-dimensional problems are considered in the majority of cases).

Example 12.1. *Suppose that a vertical shaft has been dug through the center of Earth along Earth's diameter. A small object of mass m is lowered into the shaft at Earth's surface and released without initial speed. Determine the object's speed at the center of Earth. Ignore air drag.*

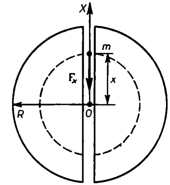

Figure 12.1

Solution. The physical system consists of the object, which can be thought of as a particle. Earth is considered being an external object. Owing to Earth's gravity the object moves in an accelerated manner toward the center. After the object has passed the center, it proceeds toward Earth's surface but in a decelerated manner. Since there is no air drag, the object reaches Earth's surface at the other end of the shaft and then, reversing its direction of motion, again begins to move toward Earth's center in an accelerated manner. Thus, here the physical phenomenon consists in the oscillatory motion of the particle. Let us apply Newton's second law. We link the inertial reference frame with Earth, place the origin at Earth's center, and direct the X axis as shown in Figure 12.1. Let us consider an arbitrary point on the X axis where

the particle is at time t; the distance from Earth's center to this point is x. When the particle is at this point, there is a force of gravity acting on it from the ball of radius x:

$$F_x = G\,\frac{mM_x}{x^2}\,, \tag{12.1}$$

where M_x is the mass of this ball. Suppose that Earth's average density is

$$\rho = \frac{M}{(4/3)\,\pi R^3}\,,$$

where $M = 6 \times 10^{24}$ kg is Earth's mass, and $R \approx 6400$ km is Earth's radius. Then $M_x = (4/3)\,\pi\rho x^3$, and the expression (12.1) for the force of gravity assumes the form

$$F_x = (4/3)\,\pi G\rho m x.$$

It can be proved that the force of gravity generated by the remaining spherical layer of thickness $(R - x)$ is zero.

Applying Newton's second law, we arrive at the differential equation describing the oscillations of the particle:

$$m\ddot{x} = -F_x, \quad \text{or} \quad \ddot{x} + (4/3)\,\pi G\rho x = 0,$$

which coincides with the differential equation describing free continuous oscillations if we put

$$\omega_0^2 = (4/3)\,\pi G\rho.$$

Thus, a particle dropped into the shaft will perform harmonic oscillations according to the law of motion

$$x = x_0 \sin(\omega_0 t + \alpha_0). \tag{12.2}$$

The amplitude x_0 and the initial phase α_0 can be found by applying the initial conditions ($x = R$ and $v = \dot{x} = 0$ at $t = 0$):

$$R = x_0 \sin \alpha_0, \; 0 = x_0\omega_0 \cos \alpha_0.$$

Hence, $\alpha_0 = \pi/2$ and $x_0 = R$. The law of motion (12.2) assumes the form

$$x = R \sin(\omega_0 t + \pi/2).$$

Knowing the law of motion, we can now determine any physical quantity characterizing the given phenomenon. Let us find the speed of the particle at Earth's center:

$$v = \dot{x} = R\omega_0 \cos(\omega_0 t + \pi/2).$$

Since at Earth's center (the origin) $x = 0$ and, hence, $\sin(\omega_0 t + \pi/2) = 0$, we have $\cos(\omega_0 t + \pi/2) = 1$. The sought speed is

$$v = R\omega_0 = \sqrt{GM/R}, \quad v = 7.8 \text{ km/s}.$$

We see that the speed is equal to the circular-orbit speed for Earth. The period of oscillations is

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{3}{4\pi G\rho}}, \quad T_0 \approx 90 \text{ min},$$

which is equal to the period of revolution of an artificial satellite around Earth in an orbit whose radius is equal to Earth's radius.

This problem can be made more complicated if we allow for air drag. Let us assume that air drag is proportional to the speed of the particle: $F_d = -r\dot{x}$. Then Newton's second law yields a differential equation for the resulting damped oscillations:

$$\ddot{x} + 2\delta\dot{x} + (4/3)\pi G\rho x = 0,$$

whose solution is

$$x = x_0 e^{-\delta t} \sin(\omega t + \alpha_0),$$

with

$$\omega = \sqrt{\frac{4}{3}\pi G\rho - \frac{r^2}{4m^2}}, \quad \delta = \frac{r}{2m}.$$

The initial amplitude x_0 and the initial phase α_0 can be found from the initial conditions ($x = R$ and $v_0 = \dot{x} = 0$ at $t = 0$):

$$R = x_0 \sin\alpha_0, \; 0 = -\delta \sin\alpha_0 + \omega\cos\alpha_0.$$

Hence,

$$x_0 = R\sqrt{1 + (\delta/\omega)^2}, \quad \alpha_0 = \tan^{-1}(\omega/\delta).$$

Consequently, here too the dynamical method has led us to the complete form of the law of motion:

$$x = R\sqrt{1+(\delta/\omega)^2}\, e^{-\delta t}\sin[\omega t + \tan^{-1}(\omega/\delta)].$$

Example 12.2. *A chunk of ice in the form of a parallelepiped of base area* $A = 1$ m^2 *and height* $H = 0.5$ m *is floating in water. The chunk is submerged to a small depth* $x_0 = 5$ cm *and then released. Find the period of oscillations. Ignore the force of resistance of water.*

Solution. The physical system consists of one object, the chunk of ice. Earth and water are the external objects. The physical phenomenon here consists in the fact that

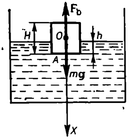

Figure 12.2

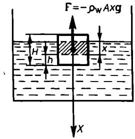

Figure 12.3

first the chunk was at rest and then it started to oscillate. In the conditions of the problem we cannot consider the chunk to be a particle, but it is easy to see that all of its points behave in the same manner. Hence, to solve the problem we need only describe the motion of one point of the chunk, say its center of mass. Let us apply Newton's second law. For the inertial reference frame we take the water (it is assumed that it remains static and that any change in its level due to the submergence of the ice can be ignored). We place the origin at the surface of the water and direct the X axis in the manner shown in Figure 12.2.

Let us consider the state of the ice parallelepiped prior to submergence. The ice is in a state of equilibrium.

Two forces act on it: the force of gravity $mg = \rho_{ice} Vg = \rho_{ice} AHg$ (where $\rho_{ice} = 900$ kg/m^3 is the density of the ice) and the buoyancy force $F_b = \rho_w Ahg$ (where $\rho_w = 10^3$ kg/m^3 is the density of the water and h is the depth of submergence in the state of equilibrium). Newton's second law yields

$$\rho_{ice} AHg - \rho_w Ahg = 0, \tag{12.3}$$

whence $h = (\rho_{ice}/\rho_w) H$.

We now wish to know what happens when the ice is submerged to an additional depth x (where x is arbitrary but small; see Figure 12.3). As a result of the additional submergence there appears an additional buoyancy force $F = \rho_w Axg = \rho_w Agx$. Combining (12.3) with Newton's second law, we arrive at the differential equation of free continuous oscillations:

$$m\ddot{x} = -\rho_w Agx, \quad \text{or} \quad \ddot{x} + \omega_0^2 x = 0 \tag{12.4}$$

where

$$\omega_0^2 = \frac{\rho_w g}{\rho_{ice} H}. \tag{12.5}$$

Equation (12.5) can be used to determine the sought quantity, the period of oscillations:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{\rho_{ice} H}{\rho_w g}}. \tag{12.6}$$

This formula yields $T_0 \approx 1.3$ s.

Equation (12.6) shows that the period of oscillations is independent of the base area A of the chunk of ice, which implies that this quantity is superfluous. The densities ρ_{ice} and ρ_w must be taken from tables.

From (12.4) and the initial conditions ($x = x_0$ and $\dot{x} = 0$ at $t = 0$) we can find the law of motion:

$$x = x_0 \sin (\omega_0 t + \pi/2).$$

Thus, the ice oscillates harmonically. In real conditions the oscillations are, of course, damped. Let us then make the problem more complicated and allow for the resistance of water. We will also change the initial conditions.

Example 12.3. *The ice from the previous example is given a push downward and thus a speed v_0 is imparted to it at the initial moment of time. Determine the speed at an arbitrary moment of time if the force of resistance of the water is proportional to the speed of the chunk, $F_r = -rv$, where r is the proportionality factor.*

Solution. Obviously, the ice will perform damped oscillations. Applying Newton's second law, we arrive at a differential equation describing these oscillations:

$$m\ddot{x} = -r\dot{x} - \rho_w A g x,$$

or

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0, \qquad (12.7)$$

where $\delta = r/2m$ is the damping factor, and $\omega_0 = \sqrt{\rho_w g/\rho_{ice} H}$ is the natural frequency.

As is known, the solution to Eq. (12.7) is given by the function

$$x = x_0 e^{-\delta t} \sin(\omega t + \alpha_0),$$

where ω is the frequency of damped oscillations: $\omega = \sqrt{\omega_0^2 - \delta^2}$.

The initial amplitude x_0 and the initial phase α_0 can be found from the initial conditions ($x = 0$ and $\dot{x} = v_0$ at $t = 0$):

$$0 = x_0 \sin \alpha_0, \quad v_0 = -x_0\delta \sin \alpha_0 > x_0\omega \cos \alpha,$$

whence $\alpha_0 = 0$ and $x_0 = v_0/\omega$.

Thus, we have the following law of motion of the chunk of ice:

$$x = \frac{v_0}{\omega} e^{-\delta t} \sin \omega t.$$

From this we can easily proceed to the formula for the speed of the ice at any arbitrary moment in time:

$$v = \dot{x} = v_0 \left(\cos \omega t - \frac{\delta}{\omega} \sin \omega t\right) e^{-\delta t}.$$

We can easily see that in the conditions of Example 12.3 the base area A is no longer superfluous and plays an important role in calculations.

Example 12.4. *The plates of a plane air capacitor are positioned vertically. A smooth dielectric shaft connects the plates horizontally, and along this there can slip a small hollow cylinder of mass $m = 10^{-3}$ kg attached to a spring whose constant is $k = 10^{-1}$ N/m (Figure 12.4). The cylinder carries an electric charge $Q = 10^{-8}$ C. An alternating voltage $U = U_0 \sin \omega t$ with $U_0 = 10^4$ V is applied to the plates. Determine at what frequency ω the amplitude of the oscillations of the cylinder will be $x_0 = 1$ cm. The distance between the plates is $d = 10$ cm. Air drag can be ignored.*

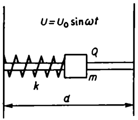

Figure 12.4

Solution. The physical system consists of a single object, the cylinder carrying an electric charge. The variable electric field existing between the plates drives the cylinder, which performs forced oscillations. Applying Newton's second law, we arrive at a differential equation for these oscillations:

$$m\ddot{x} = -kx + \frac{Q}{d} U_0 \sin \omega t, \quad \text{or} \quad \ddot{x} + \omega_0^2 x = \frac{QU_0}{md} \sin \omega t.$$

The solution to this equation is given by the function

$$x = x_0 \sin \omega t, \tag{12.8}$$

$$x_0 = \frac{QU_0}{(\omega_0^2 - \omega^2)\, md}. \tag{12.9}$$

The law of motion (12.8) implies that the cylinder oscillates harmonically. The sought frequency can be found by solving Eq. (12.9):

$$\omega = \sqrt{\frac{k}{m} - \frac{QU_0}{mdx_0}},$$

which yields $\omega \approx 9.5$ rad/s.

We have studied several examples involving mechanical oscillations. All have been solved by the same dy-

namical method. Thus, mechanical oscillation problems constitute a particular case of the basic problems of dynamics.

13. Conservation Laws

In addition to the kinematic and dynamical methods for solving physics problems there is one more method, possibly a more important one, the *method of applying conservation laws*. This method is more universal than the first two. While the use of the kinematic and dynamical methods is restricted to classical physical systems, the conservation-law method can be applied to both classical and quantum systems.

It must be noted, however, that when applied to classical physical systems the kinematic and dynamical methods are more general than the conservation-law method. This is especially true of mechanical systems. In principle any formulated mechanical problem can be physically solved via the kinematic and dynamical methods, or simply the kinematic-dynamical method. The same cannot be said of the conservation-law method: far from all mechanical systems can be solved by applying the conservation-law method. But in the more complicated systems the conservation-law method sometimes leads to a result faster than the kinematic-dynamical method.

We have already noted that there is no one universal method for solving physics problems. What is important is a system of methods. Therefore, it is meaningless to contrast one method with another: each has its strong and weak points. Nature is so diversified in its properties and manifestations that to reveal the various relationships in physical phenomena we need an intelligent combination of various methods. Hence, in solving physics problems it is also advisable to use a system of methods, including the kinematic-dynamical method and the conservation-law method.

The method now to be considered is based on a group of conservation laws. There are quite a number of such laws in physics. The following four are used for classical systems: the *law of momentum conservation*, the *law of*

energy conservation (for mechanical systems its particular case is employed, the *law of energy conservation in mechanics*), the *law of angular momentum conservation*, and the *law of electric charge conservation*. What is common to all four laws is the statement that a certain physical quantity is conserved in certain conditions. If the conserved physical quantity is denoted by A and the set of conditions in which the particular law holds true by B, conservation laws can be formulated in a generalized manner thus:

> if conditions B are met, then $A = \text{const}$, or in equivalent form, if conditions B are met, then $\Delta A = 0$, where ΔA is a variation of A.

In the majority of cases conservation laws are applied if the objects in the system interact. Three stages must be distinguished in such interaction: the first is the state of the objects prior to interaction, the second is the interaction itself, and the third is the state of the objects after interaction. The interaction process is unimportant to conservation laws. What is important is that the value of the corresponding physical quantity does not change as a result of such interaction (i.e. its values before and after the interaction must be equal). Therefore, the conservation-law method consists of the following steps:

(1) establish what objects are included in the physical system;

(2) check to see whether conditions B are met;

(3) select an inertial reference frame (with respect to which we will subsequently find the values of A);

(4) find the value of A prior to interaction (we denote this by A_1);

(5) determine the value of A after interaction (we denote this by A_2);

(6) write the conservation law in the form $A_1 = A_2$ or in the form $\Delta A = 0$ $(A_2 - A_1 = 0)$; and

(7) if the law is a vector one, it is usually "projected" onto the coordinate axes; the result is three equations, $A_{1x} = A_{2x}$, $A_{1y} = A_{2y}$, and $A_{1z} = A_{2z}$, which are equivalent to the vector equation.

In this section we consider only two laws: the law of momentum conservation and the law of energy conservation in mechanics. The other laws will be discussed later.

Example 13.1. *The inelastic collision. Two objects with masses* $m_1 = 2$ kg *and* $m_2 = 3$ kg *that have been moving with velocities* $\mathbf{v}_1 = (3\mathbf{i} + 4\mathbf{j})$ *and* $\mathbf{v}_2 = (-2\mathbf{i} + 3\mathbf{j})$ *with respect to a certain inertial reference frame collide inelastically. Find their velocity* $\mathbf{v}$ *after collision. The effect of other objects can be ignored.*

Solution. The physical system incorporates two objects, m_1 and m_2. Since the terms of the problem allow us to neglect the effect of external objects, the system is closed. Note that the laws of motion cannot be established (if we use the kinematic method) unless we know the initial conditions (i.e. the position of the objects at $t = 0$). The physical phenomenon here is the inelastic interaction of the two objects constituting the closed system. Given the masses and the velocities of the objects prior to interaction, find the velocity of the objects after interaction.

We apply the momentum conservation law. The possibility of applying this law has been verified. The inertial reference frame has been selected in the hypothesis. Let us determine the momentum of each object prior to interaction and find the vector sum of the momenta: $\mathbf{p}_1 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$. Next we find the momentum of the system after interaction (as a result of the inelastic collision the objects stick together and move with a common velocity $\mathbf{v}$): $\mathbf{p}_2 = (m_1 + m_2)\,\mathbf{v}$. By the law of momentum conservation,

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\,\mathbf{v},$$

whence

$$\mathbf{v} = \frac{m_1}{m_1+m_2}\,\mathbf{v}_1 + \frac{m_2}{m_1+m_2}\,\mathbf{v}_2.$$

Projecting this vector equation on the coordinate axes, we find the components of the sought velocity vector:

$$v_x = \frac{m_1}{m_1+m_2}\,v_{1x} + \frac{m_2}{m_1+m_2}\,v_{2x}, \quad v_x = 0;$$

$$v_y = \frac{m_1}{m_1+m_2}\,v_{1y} + \frac{m_2}{m_1+m_2}\,v_{2y}, \quad v_y = \frac{17}{5}\ \text{m/s}.$$

Thus, the two objects will move along the Y axis with a speed $v_y = 17/5$ m/s.

Sometimes the chosen physical system as a whole is not closed and, hence, the momentum conservation law cannot be applied. But the system may prove to be closed along a certain direction (say, along the X axis); in other words, the algebraic sum of the projection of external forces on this direction is zero. Then we can write the momentum conservation law (only for this direction) in the form

$$p_{1x} = p_{2x}.$$

Example 13.2. *A cart with sand whose combined mass M is 100 kg is moving in a straight line and uniformly along a horizontal surface with a speed $v_0 = 3$ m/s (Figure 13.1). A ball of mass $m = 20$ kg falls onto the cart from*

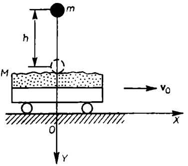

Figure 13.1

a height $h = 10$ m reckoned from the surface of the sand (the initial speed of the ball is zero). Determine the speed of the cart-sand-ball system after the interaction. Friction can be neglected.

Solution. The physical system consists of the cart with sand (considered as a single object) and the ball. It is not closed, since prior to interaction Earth's gravity acted on the ball and this force was not balanced by any other external force. So, generally speaking, the momentum conservation law cannot be applied. But in

the direction in which the cart is moving no external forces act. Hence, in this direction the momentum conservation law can be applied. We link the inertial reference frame with Earth and direct the coordinate axes as shown in Figure 13.1. The component of the momentum vector $\mathbf{p}_1$ of the system along the X axis was $p_{1x} = Mv_0$ prior to interaction; the same component after interaction was $p_{2x} = (M + m)\,v$, with v the sought speed. By momentum conservation,

$$Mv_0 = (M + m)\,v,$$

whence

$$v = \frac{Mv_0}{M+m}, \tag{13.1}$$

which yields, after we substitute numerical values, $v = 2.5$ m/s. Equation (13.1) shows that the sought speed does not depend on h and, hence, the data on h is superfluous.

We could include Earth as an object comprising the physical system. Then the system of the three objects will be closed. Since Earth now belongs to the physical system and owing to the force of gravity generated by the ball will move in an accelerated manner with respect to an inertial reference frame, we cannot link an inertial reference frame with Earth, strictly speaking. But it can easily be demonstrated that Earth's speed and acceleration (in conditions of this and similar problems, where the masses of objects are extremely small if compared to Earth's mass) at any moment in time are so small that they can be ignored and Earth can be considered as a fixed object.

Let us find, for instance, the speed that Earth will gain as a result of interaction with the ball (the maximal value of the speed which Earth could gain in the conditions of the problem). Very often in physics an inertial reference frame is selected as linked to the center of mass or center of inertia of the system. In what follows we denote this reference frame as the center-of-mass (CM) reference frame, or the center-of-inertia (CI) reference frame. By the center of mass of a system we mean the point whose radius vector $\mathbf{r}_{\mathrm{CM}}$ is determined from the

equation

$$\mathbf{r}_{\mathrm{CM}}=\frac{\sum m_i r_i}{\sum m_i}\ . \qquad (13.2)$$

It can be demonstrated that the *CM of a system moves like a particle whose mass is equal to the total mass of the system, while the acting force is equal to the geometric sum of all the external forces acting on the system* (the theorem on the motion of center of mass). Let us write the equation of motion of the center of mass:

$$m\,\frac{d\mathbf{V}_{\mathrm{CM}}}{dt}=\sum \mathbf{F}_i,$$

where $m=\sum m_i$ is the total mass of the system, $\mathbf{V}_{\mathrm{CM}}$ the velocity vector of the CM, and $\sum \mathbf{F}_i$ the vector sum of the external forces.

If the physical system is closed, then $\sum \mathbf{F}_i=0$ and $\mathbf{V}_{\mathrm{CM}}=$ const, that is, the center of mass of a closed system moves uniformly and in a straight line. Hence, the reference frame linked with the center of mass of such a system is inertial. Since in the CM reference frame the origin coincides with the center of mass, $\mathbf{r}_{\mathrm{CM}}=0$, and (13.2) yields

$$\sum m_i\mathbf{r}_i=0. \qquad (13.3)$$

Differentiating (13.3) with respect to time t, we get

$$\sum m_i\mathbf{v}_i=0, \qquad (13.4)$$

that is, the momentum of a closed system with respect to the CM reference frame is zero at any moment in time. Let us apply this result to calculate the increase in Earth's speed as the result of interaction between Earth and the ball (Figure 13.2; the origin of the CM reference frame, O, is shifted slightly to the right). Equation (13.4) yields

$$Mv_{\mathrm{E}}-mv_{\mathrm{ball}}=0, \qquad (13.5)$$

where M is Earth's mass, v_{E} Earth's speed (more exactly, the increment of the speed due to the interaction between Earth and the ball), m the mass of the ball, and v_{ball}

the speed of the ball. Equation (13.5) can be used to find the sought speed:

$$v_E = \frac{m}{M} v_{ball} = \frac{m}{M}\sqrt{2gh}, \quad v_E \approx 5 \times 10^{-23} \text{ m/s}.$$

This speed is fantastically small. Moving at such a speed, it would take Earth 6×10^{12} years to shift by a distance of 1 cm. In what follows, when studying the motion of

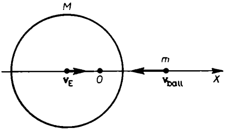

Figure 13.2

objects whose mass is small compared with that of Earth, we ignore the interaction between Earth and such objects and assume Earth to be stationary.

The law of energy conservation in mechanics is linked to the notions of *kinetic*, E_k, and *potential*, E_p, *energies*. Another extremely important concept is that of *work*, W. As is known, a force $\mathbf{F}$ performs an amount of work over an elementary displacement $d\mathbf{r}$ equal to

$$dW = \mathbf{F} \cdot d\mathbf{r}. \qquad (13.6)$$

The work performed by force $\mathbf{F}$ over a path l is expressed by the integral

$$W = \int_l \mathbf{F} \cdot d\mathbf{r}, \qquad (13.7)$$

where the integral is evaluated along curve l.

There are cases where we have to know the work performed in rectilinear motion. Since $d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$, we can represent (13.6) in the form

$$dW = \mathbf{F} \cdot \mathbf{i}\ dx + \mathbf{F} \cdot \mathbf{j}\ dy + \mathbf{F} \cdot \mathbf{k}\ dz$$
$$= F\ dx \cos\alpha_1 + F\ dy \cos\alpha_2 + F\ dz \cos\alpha_3,$$

where α_1, α_2, and α_3 are the angles that the force (vector) **F** forms, respectively, with the unit vectors **i**, **j**, and **k** pointing along the X, Y, and Z axes. If the motion is along a straight line (say, the X axis),

$$\mathrm{d}W = F\,\mathrm{d}x \cos \alpha.$$

The work performed by force F along the segment from x_1 to x_2 in this case is determined by the formula

$$W = \int_{x_1}^{x_2} F \cos \alpha \, \mathrm{d}x.$$

If the force is constant, we have no difficulty in calculating the work. Often the DI method is used when a calculation is done of the work performed by a variable force (see Section 6). Let us restrict our discussion to the rectilinear case and assume that $|\cos \alpha| = 1$. A force may depend on the x coordinate (in the general case, on the y and z coordinates also), on the components of the velocity $v_x = v$ (in the general case on the other components of **v** as well), and on time t.

Let us start with the case where the force depends on position, or $F = F(x)$. The elementary work done by such a force is

$$\mathrm{d}W = F(x)\,\mathrm{d}x.$$

The work performed on the segment from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,\mathrm{d}x.$$

Example 13.3. *First an object is lifted from the bottom of a shaft of depth $h_1 = R/2$ (R is Earth's radius) to Earth's surface and then it is lifted still higher, to an altitude $h_2 = h_1 = R/2$ above Earth's surface. In which case is the work done greater?*

Solution. We can easily see that this problem involves an estimate. To answer the question posed by the problem, let us find the ratio W_1/W_2, where W_1 is the work done in the first case and W_2 in the second. In both the work is done against the force of gravity, but the laws describing these forces are different. Exampe 12.1 showed

that the force of gravity in the first case is

$$F_1 = (4/3)\,\pi G\rho m x,$$

while in the second it is

$$F_2 = GmM/x^2.$$

The variation of these forces is illustrated in Figure 13.3. Thus, the forces are variable, and to calculate W_1 and

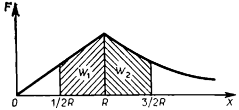

Figure 13.3

W_2 we must apply the DI method. The elementary work done over dx are

$$\mathrm{d}W_1 = F_1(x)\,\mathrm{d}x \quad \text{and} \quad \mathrm{d}W_2 = F_2(x)\,\mathrm{d}x.$$

Upon integrating within the appropriate limits we get

$$W_1 = \int_{R/2}^{R} \frac{4}{3}\,\pi G\rho m x\,\mathrm{d}x = \frac{3}{8}\,\frac{GmM}{R},$$

$$W_2 = \int_{R}^{(3/2)R} \frac{GmM}{x^2}\,\mathrm{d}x = \frac{GmM}{3R}$$

and, hence,

$$W_1/W_2 = 9/8, \quad \text{i.e.} \quad W_1 > W_2.$$

A force may depend on a component of the velocity, $v_x = v$. When calculating the work done by such a force (or against it), we must find the law of variation of v with time t, that is, solve a basic problem of dynamics by employing Newton's second law. In this case the elementary work done by the force is

$$\mathrm{d}W = F(v)\,\mathrm{d}x = F(v)\,v\,\mathrm{d}t. \qquad (13.8)$$

By Newton's second law,

$$m\frac{dv}{dt} = F(v) + \sum F_i, \tag{13.9}$$

where $\Sigma\ F_i$ stands for the algebraic sum of the projections on the direction of motion of other forces acting on the given object. Solving Eq. (13.9) and allowing for the initial conditions, we find the law of speed variation with time: $v = v(t)$. Substitution of this law into (13.8) and integration yield a formula for work:

$$W = \int_{t_1}^{t_2} F(t)\,v(t)\,dt. \tag{13.10}$$

Example 13.4. *Calculate the air drag on the skydiver of Example 11.3 in the first three seconds and in the first thirty seconds.*

Solution. Since air drag depends on speed ($F_d = -kv$) and the law (11.19) of speed variation has been found, formula (13.10) gives us the amount of work done against air drag:

$$W = \int_0^t kv(t)\,v(t)\,dt = \frac{m^2g^2}{k}\int_0^t (1 - e^{-(k/m)t})^2\,dt$$
$$= \frac{m^2g^2}{k}\left[t + \frac{2m}{k}(e^{-(k/m)t} - 1) - \frac{m}{2k}(e^{-2(k/m)t} - 1)\right]. \tag{13.11}$$

Substituting the values $t_1 = 3$ s and $t_2 = 30$ s, we get

$$W_1 \approx 10^4\ \text{J} \quad \text{and} \quad W_2 \approx 1.1 \times 10^6\ \text{J}.$$

Next we find the ratio of the work:

$$W_2/W_1 = 110.$$

Hence, the work done against air drag increased by a factor of 110 when time increased only 10-fold. This can be explained by the growth in air drag and the speed of fall.

In conclusion we investigate the case where the force is time-dependent: $F = F(t)$. Here, too, to find the law of variation of speed on time, $v = v(t)$, we must solve

a basic problem of dynamics. The elementary work done by the force is

$$dW = F(t)\,dx = F(t)\,v(t)\,dt.$$

After finding the law of variation of the speed, we can write the formula for the work in the following form:

$$W = \int_{t_1}^{t_2} F(t)\,v(t)\,dt. \qquad (13.12)$$

Example 13.5. *Determine the work done by the thrust of the braking engine in Example 11.4 in the first second.*

Solution. Since the thrust of the braking engine depends on time ($F = kt$), the law (11.27) of speed variation has been found, and duration of braking is known, we apply formula (13.12) and get

$$W = \int_0^t kt\left(v_0 - \frac{kt^2}{2m}\right)dt = \frac{kv_0t^2}{2} - \frac{k^2t^4}{8m},$$

$$W \approx 2.5 \times 10^6\ \mathrm{J}. \qquad (13.13)$$

Sometimes the work can be calculated by using the theorem on the change of the kinetic energy of a physical system consisting of particles. By this theorem the work of all the forces acting on such a system is equal to the variation in the kinetic energy of the system:

$$W = \Delta E_k. \qquad (13.14)$$

In Example 11.4, two of the three forces acting on the object balance each other. The remaining (unbalanced) force is the thrust produced by the braking engine whose work we have set out to find. Hence, W in (13.14) is the work done by the thrust produced by the braking engine, and $\Delta E_k = mv_0^2/2 - mv^2/2$. Applying formula (13.14) and allo ing for the law (11.27) of speed variation, we get

$$W = \frac{mv_0^2}{2} - \frac{m}{2}\left(v_0 - \frac{kt^2}{2m}\right)^2 = \frac{kv_0t^2}{2} - \frac{k^2t^4}{8m},$$

$$W \approx 2.5 \times 10^6\ \mathrm{J},$$

which coincides with the result obtained earlier via the DI method (see (13.13)).

By the law of energy conservation in mechanics, the *total mechanical energy of a closed system in which only conservative forces act is a constant*:

$$E = E_k + E_p = \text{const}, \quad \text{or} \quad \Delta(E_k + E_p) = 0. \qquad (13.15)$$

If there are nonconservative forces in a closed system, the system's total mechanical energy is not conserved and its variation is equal to the work done by the nonconservative forces:

$$\Delta E = W_{dis}, \qquad (13.16)$$

where W_{dis} is the work done by the dissipative (or nonconservative) forces.

Example 13.6. *Determine the velocity that a meteorite of mass m has at a distance* $r = 1.5 \times 10^{11}$ m *from the Sun (mass M) if at infinity it had a zero velocity and is moving toward the Sun. The effect of all other objects can be neglected.*

Solution. The physical system consists of two bodies, the meteorite and the Sun. The meteorite can be thought of as being a particle, while the Sun is assumed to be a solid ball of radius $R = 7 \times 10^5$ km. The physical phenomenon associated with this system consists in the meteorite moving toward the Sun under the Sun's gravity. Given the initial state of the physical system, we must determine one of the parameters of the meteorite's motion (the velocity v) in the finite state. This constitutes a basic problem of dynamics, which could be solved via the dynamical method by applying Newton's second law. But here there is no need to find the v vs. t dependence, since what we are looking for is the value of v in the final state; in other words, there is no need to describe the entire process of the meteorite's motion. Hence, it is advisable to employ the law of energy conservation in mechanics.

The chosen system of bodies is closed (by hypothesis the effect of other bodies can be ignored). Only conservative gravitational forces act in the system. We select the inertial reference frame as the one linked with the

Sun (we assume the Sun to be fixed; see Example 13.2). The total mechanical energy E_1 of the system at the beginning of the interaction of the bodies is zero (their kinetic energies are zero, and assuming the initial position of the system to be the zero position we can set the initial potential energy at zero, too). Let us determine the total

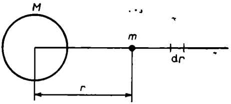

Figure 13.4

mechanical energy of the system at the end of the interaction, E_2, when the meteorite is at a distance $r = 1.5 \times 10^{11}$ m from the Sun (Figure 13.4). It consists of the meteorite's kinetic energy $E_k = mv^2/2$ and its potential energy. The latter is determined by the work performed by the force of gravity in the course of the meteorite's movement from the final to the initial position.

Since the force of gravity depends on distance r, that is, it constitutes a variable force, we can employ the DI method to calculate the work performed by this force. Let us divide the entire path of the meteorite into intervals so small that in each such interval of length dr we can ignore the variation in force of gravity, assuming it to be constant. Then the elementary work on such an interval is

$$dW = G\frac{mM}{r^2}\cos\alpha\, dr = -G\frac{mM}{r^2}\, dr.$$

Summing the elementary work done on each interval, we get the total work W, which gives the value of the mutual potential energy of the system E_p:

$$W = E_p = -\int_r^\infty G\frac{mM}{r^2}\, dr = -GmM\left(-\frac{1}{r}\right)\Big|_0^\infty = -\frac{GmM}{r}.$$

Thus, the total mechanical energy of the system in the initial state, E_1, is zero, and in the final state $E_2 = mv^2/2 - GmM/r$. By the law of energy conservation in mechanics,

$$0 = \frac{mv^2}{2} - \frac{GmM}{r} .$$

Solving this equation for v, we find the sought velocity:

$$v = \sqrt{2GM/r}, \quad v \approx 42.2 \text{ km/s},$$

where the values of the gravitational constant G and the Sun's mass M were taken from tables.

Example 13.7. *A small steel cube of mass $M = 1$ kg is at rest on a horizontal surface. A small steel ball of mass*

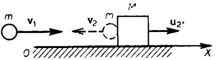

Figure 13.5

$m = 10$ g *is flying at the cube with a velocity $v_1 = 10^3$ m/s, hits it, and bounces off elastically in the opposite direction (see Figure 13.5). Find the distance that the cube will travel before it stops if the coefficient of friction between the horizontal surface and the cube is $k = 0.2$.*

Solution. Two bodies constitute the physical system: the cube and the ball. These may be considered as particles. Earth is taken as an external object. The physical phenomenon consists in the elastic interaction of ball and cube (the interaction with the external object, Earth, can be ignored in their subsequent motion. The initial state of the system (prior to interaction) is known. What is sought is one of the parameters of the cube's motion (the distance the cube travels after impact).

Since the forces that emerge as the result of interaction of cube and ball are unknown, there is no way in which we can employ the dynamical method to describe the process. Let us apply the laws of momentum and

energy conservation in mechanics. On the whole the system is not closed, but in the direction in which the ball moves it can be considered closed. We select the inertial reference frame linked with Earth and point the X axis as shown in Figure 13.5. Prior to the interaction the momentum of the system is $\mathbf{p}_1 = m\mathbf{v}_1$. The momentum of the system after interaction is $\mathbf{p}_2 = m\mathbf{v}_2 + M\mathbf{u}_{21}$, where $\mathbf{v}_2$ and $\mathbf{u}_{21}$ are the velocity vectors of the cube and the ball, respectively, after interaction. By momentum conservation,

$$m\mathbf{v}_1 = m\mathbf{v}_2 + M\mathbf{u}_{21}.$$

Projecting this vector equation on the X axis, we get

$$mv_1 = -mv_2 + Mu_{21}.$$

By energy conservation in mechanics,

$$mv_1^2/2 = mv_2^2/2 + Mu_{21}^2/2.$$

Allowing for the fact that $m \ll M$ and solving the above system of equations, we find that

$$v_2 \approx v_1, \quad u_{21} = \frac{2m}{M+m} v_1 \approx \frac{2m}{M} v_1.$$

Let us now study the motion of the cube after impact. The statement of the new problem is obvious: *a cube of mass $M = 1$ kg having an initial velocity u_{21} skids along the horizontal surface (the friction coefficient f is equal to 2×10^{-1}) and finally stops; find the distance it has traveled.*

Two bodies constitute the physical system, the cube and Earth. The physical phenomenon consists in the cube moving in a decelerated manner as a result of its interaction with Earth. The initial configuration of the system is known. We must find one of the parameters of the cube's motion (distance of travel). This constitute a basic problem of dynamics. Since the forces of interaction of the cube with Earth are known, the problem can be solved either by the dynamical method or by the conservation-law method. Applying Newton's second law,

$$Ma = fMg,$$

we find the acceleration. Solving an inverse problem of kinematics, we find the distance l_1 the cube traveled before it stopped:

$$l_1 = \frac{u_{21}^2}{2fg} = \frac{2m^2v_1^2}{fgM^2}, \quad l_1 \approx 100 \text{ m}. \qquad (13.17)$$

Let us take the alternative approach. We wish to solve the problem via the conservation-law method (in mechanics). The selected system is closed, but the law of energy conservation in the form (13.15) cannot be applied (there is a nonconservative friction force $F_{fr} = fMg$). Assuming that the internal nonconservative friction force is an external one, we find that Eq. (13.16) yields

$$\frac{Mu_{21}^2}{2} = fMgl_1.$$

This leads us to a result that coincides with (13.17), which was arrived at via the dynamical method:

$$l_1 = \frac{u_{21}^2}{2fg} = \frac{2m^2v_1^2}{fgM^2}, \quad l_1 \approx 100 \text{ m}.$$

The solved problem could have also been formulated in, say, the following manner (a situation problem): *as a result of what interaction of the ball and the cube will the length of the path traveled by the cube be maximal?*

Let us change the terms of Example 13.7, namely, *we assume the cube to be an inelastic body, with all other conditions remaining unchanged. Find the distance that the cube travels before it stops.*

The interaction process (inelastic collision) is described by the law of momentum conservation:

$$mv_1 = (m + M)u_{22}.$$

This gives us the initial velocity of the cube (we allow for the fact that $m \ll M$):

$$u_{22} \approx mv_1/M.$$

Solving the dynamical problem of the cube's motion after impact (by either of the two methods), we find the distance of travel:

$$l_2 = \frac{u_{22}^2}{2fg} = \frac{m^2v_1^2}{2fgM^2}, \quad l_2 \approx 25 \text{ m}. \qquad (13.18)$$

Equations (13.17) and (13.18) show that in the case of an elastic collision the distance traveled by the cube after impact is four times the distance in the case of an inelastic collision.

Let us change the terms of Example 13.7: *suppose that as the result of collision with the cube the ball pierces the cube and continues its motion in the same direction with a velocity* $v_2 = 500$ m/s, *while all other conditions remain the same. Find the distance the cube travels before stopping.*

We again apply the momentum conservation law and get

$$mv_1 = mv_2 + Mu_{23},$$

which yields the initial velocity of the cube at

$$u_{23} = m(v_1 - v_2)/M.$$

The distance the cube travels before stopping is

$$l_3 = \frac{u_{23}^2}{2fg} = \frac{m^2(v_1 - v_2)^2}{2fgM^2}, \quad l_3 \approx 6.25 \text{ m}.$$

Thus, in the case of an elastic collision (with $m \ll M$) both the velocity u_{21} and the distance l_1 traveled by the cube before stopping are the maximum possible values.

The dependence of the initial velocity and the distance traveled by the cube before stopping on the m/M ratio can also be studied. For instance, for any mass ratio

m/M (but $m \ll M$) we have

$$u_{21}/u_{22} = 2$$

and, hence,

$$l_1/l_2 = 4$$

Chapter 4

THE MOTION OF A RIGID BODY

14. Rigid-Body Dynamics

The acceleration $\mathbf{a}_{CM}$ of the center of mass of a rigid body is determined by the theorem on the motion of the center of mass:

$$m\mathbf{a}_{CM} = \sum \mathbf{F}, \qquad (14.1)$$

where m is the mass, and $\sum \mathbf{F}$ stands for the vector sum of all the external forces acting on the body.

The form of Eq. (14.1) coincides with Newton's second law for a particle, (11.4), and, hence, the method of applying this law consists of the same operations. The vector equation (14.1) is equivalent to the following three equations:

$$ma_{\mathrm{CM}x} = \sum F_x, \quad ma_{\mathrm{CM}y} = \sum F_y, \quad ma_{\mathrm{CM}z} = \sum F_z. \qquad (14.2)$$

For a particle and, hence, for a rigid body we have the following equation of motion:

$$\dot{\mathbf{L}} = \sum M, \qquad (14.3)$$

where $\dot{\mathbf{L}} = \mathrm{d}\mathbf{L}/\mathrm{d}t$ (vector $\mathbf{L}$ is defined below), and $\Sigma\mathbf{M}$ is the vector sum of the moments of the external forces (these moments are also called torques) about a fixed point O.

If point O is taken as the origin of a Cartesian coordinate system, then, as usual the vector equation (14.3) is equivalent to the following three equations:

$$\frac{\mathrm{d}L_x}{\mathrm{d}t} = \sum M_x, \quad \frac{\mathrm{d}L_y}{\mathrm{d}t} = \sum M_y, \quad \frac{\mathrm{d}L_z}{\mathrm{d}t} = \sum M_z, \qquad (14.4)$$

where L_x, L_y, and L_z are the projections of the angular momentum vector $\mathbf{L}$ on the coordinate axes. They are known as the *angular momenta of a rigid body about the fixed X, Y, and Z axes*, respectively. It can be shown that for a particle and a rigid body the following equations hold true:

$$L_x = J_x\omega_x, \quad L_y = J_y\omega_y, \quad L_z = J_z\omega_z, \qquad (14.5)$$

where J_x, J_y, and J_z are the moments of inertia of a particle or rigid body about the X, Y, and Z axes, respectively, and ω_x, ω_y, and ω_z are the projections of the angular velocity ω on the same axes.

If we allow for (14.5), we can rewrite Eqs. (14.4) as

$$\frac{\mathrm{d}(J_x\omega_x)}{\mathrm{d}t} = M_x, \quad \frac{\mathrm{d}(J_y\omega_y)}{\mathrm{d}t} = M_y, \quad \frac{\mathrm{d}(J_z\omega_z)}{\mathrm{d}t} = M_z. \qquad (14.6)$$

If the moments of inertia J_x, J_y, and J_z are constant, the equations of motion assume the form

$$J_x\beta_x = M_x, \quad J_y\beta_y = M_y, \quad J_z\beta_z = M_z, \tag{14.7}$$

where $\beta_x = d\omega_x/dt$, $\beta_y = d\omega_y/dt$, and $\beta_z = d\omega_z/dt$ are the projections of the angular momentum vector β on the coordinate axes. These equations are known as the *equations of motion with respect to the fixed X, Y, and Z axes*, respectively.

A rigid body has six degrees of freedom; hence, we need six independent equations to describe its motion. These are either the two vector equations (14.1) and (14.3) or the equivalent system of six equations (14.2) and (14.6). The method of applying the laws (14.2) differ in no way from that of applying Newton's second law. The method of applying the laws (14.6) also closely resembles the method of applying Newton's second law if two additional operations are introduced: finding the moments of inertia of bodies and the moment (or torque) of the external forces about the appropriate axes. Thus, for a rigid body the dynamical method remains practically the same as for a particle.

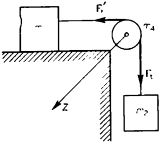

Figure 14.1

Example 14.1. *Let us consider the simplified version of Example 11.2 and assume that the pulley is a solid cylinder of radius R = 10 cm with a mass m_4 = 8 kg. Determine the system's acceleration and the tensile stress developed by the string.*

Solution. The same four bodies constitute the physical system: the two blocks with masses m_1 and m_2, the string, and the pulley (Figure 14.1). But now the pulley is not only a body whose mass must be taken into account. We must allow for its dimensions, i.e. we cannot assume that it is a particle any more. One assumption is

that the pulley is a rigid body. Its center of mass is fixed, and the pulley rotates about a fixed axis, the Z axis, which passes through the pulley's center of mass. We apply Eq. (14.7) with respect to the Z axis to the pulley. An inertial reference frame has been selected. Two unequal tensile stresses act on the pulley: $F''_t \neq F'_t$. All the other forces acting on the pulley compensate each other. The moments of forces F'_t and F''_t about the Z axis are $M'_z = -F'_t R$ and $M''_z = F''_t R$. The moment of inertia of the pulley (a solid cylinder) about the same axis is $J_z = (1/2)\, m_4 R^2$. In what follows the subscripts standing for the axes at the moments of forces, angular momenta, and other quantities will be dropped. Employing the equation of motion (14.7), we get

$$(1/2)\, m_4 R^2 \beta = (F''_t - F'_t)\, R.$$

Applying Newton's second law to particles m_1 and m_2, we find that

$$m_1 a = F'_t, \quad m_2 a = m_2 g - F''_t.$$

The equation that links the linear acceleration a with the angular acceleration β,

$$\beta = a/R,$$

completes the system of equations. Solving the system, we get

$$a = \frac{m_2}{m_1 + m_2 + (1/2)\, m_4}\, g, \qquad a \approx 6.5\ \text{m/s}^2;$$

$$F'_t = \frac{m_1 m_2}{m_1 + m_2 + (1/2)\, m_4}\, g, \qquad F'_t \approx 6.5\ \text{N};$$

$$F''_t = \left(1 - \frac{m_2}{m_1 + m_2 + (1/2)\, m_4}\right) m_2 g, \quad F''_t \approx 33\ \text{N};$$

$$\beta = \frac{m_2/R}{m_1 + m_2 + (1/2)\, m_4}\, g, \qquad \beta \approx 65\ \text{rad/s}^2.$$

We see that the acceleration a of the system has diminished considerably. It is also interesting to note how different the tensile stresses developed by the string now are: the tensile stress F''_t must be considerably higher than F'_t since the moments of these forces have different signs.

Example 14.2. *One end of a hanging string is tied to a support at point O (Figure 14.2), while the other is wound around a solid narrow cylinder (disk) of mass* $m = 10$ kg *and radius* $R = 10$ cm. *Determine the acceleration of the disk's center of mass and the tensile stress developed by the string, which is massless and nonextendable.*

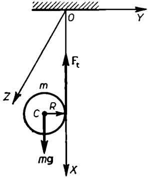

Figure 14.2

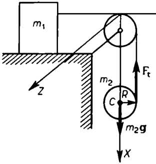

Figure 14.3

Solution. Two bodies, cylinder and string, constitute the physical system. The cylinder cannot be thought of as being a particle. We assume it to be a rigid body. Its center of mass (point C) moves downward, and it itself rotates about a moving axis passing through the center of mass. Let us employ the theorem on the motion of the center of mass, (14.1), and the equation of motion, (14.7). We link the inertial reference frame with Earth and point the coordinate axes as shown in Figure 14.2. Two forces act on the cylinder, the force of gravity $m\mathbf{g}$ and the tensile stress $\mathbf{F}_t$. By the theorem on the motion of the center of mass,

$$m\mathbf{a}_{CM} = m\mathbf{g} + \mathbf{F}_t.$$

Projecting this vector equation on the X axis, we get

$$ma_{CM} = mg - \mathbf{F}_t.$$

The cylinder moves with respect to the moving axis, but this axis is also in translational motion; in this case

the equation of motion (14.7) remains valid:

$$\frac{1}{2} mR^2\beta = F_t R,$$

where $\beta = a_{CM}/R$ is the angular acceleration. Solving the system of equations, we get

$$a_{CM} = \frac{2}{3} g, \quad F_t = \frac{5}{3} mg.$$

Hence, $a_{CM} \approx 6.6$ m/s^2 and $F_t \approx 163$ N.

Let us make the problem more complicated: *suppose that a block (particle) of mass $m_1 = 1$ kg is attached to the upper end of the string, the block can move (skid) without friction along a horizontal surface, and the string is swung over a massless pulley (Figure 14.3).*

We denote the mass of the cylinder by m_2. Applying the dynamical method, we set up a closed system of equations for the translational motion of block and cylinder, respectively,

$$m_1 a_1 = F_t, \quad m_2 a_{CM} = m_2 g - F_t,$$

and for the rotational motion of the cylinder,

$$\frac{1}{2} m_2 R^2 \beta = F_t R.$$

The acceleration a_{CM} of the cylinder's center of mass, the acceleration a_1 of the block (particle), and the angular acceleration β, are linked through the following relationship:

$$a_{CM} = a_1 + \beta R.$$

Solving the obtained system of equations, we get

$$a_{CM} = \frac{2+m_2/m_1}{3+m_2/m_1} g, \quad a_1 = \frac{m_2/m_1}{3+m_2/m_1} g,$$

$$F_t = \frac{m^2}{3+m_2/m_1} g, \qquad \beta = \frac{2+m_2/m_1}{R(3+m_2/m_1)} g.$$

Substituting the numerical values, we get

$$a_{CM} \approx 9.1 \text{ m/s}^2, \quad a_1 \approx 7.5 \text{ m/s}^2, \quad F_t \approx 7.5 \text{ N},$$
$$\beta \approx 15.1 \text{ rad/s}^2.$$

The problem can be made still more complicated by allowing for friction between block m_1 and the horizontal surface, by taking into account the pulley's mass, by assuming that the pulley is a rigid body, and the like. All such problems can be solved by the same dynamical method.

Let us drastically change the terms of Example 14.2 by introducing a friction force, static friction. This force is inevitable in the problem about to be discussed and cannot be ignored.

Example 14.3. *A massless and nonextendable string is wound around a solid cylinder (disk) of mass* $m = 10$ kg *and radius* $R = 10$ cm. *The cylinder can move without slippage along a horizontal surface. A constant horizontal force* $F = 30$ N *is applied to the free end of the string (Figure 14.4). Find the acceleration of the center of mass.*

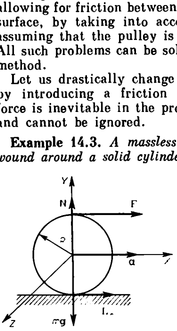

Figure 14.4

Solution. The physical system consists of a single rigid body, the cylinder, whose center of mass moves rectilinearly. The cylinder also rotates about a moving axis whose direction of motion does not vary. We are looking for the acceleration of the center of mass. This constitutes the basic problem of rigid-body dynamics. Let us apply the dynamical method. The inertial reference frame is linked with Earth, the X axis points to the right, and the rotation axis is parallel to the Z axis. Four forces act on the cylinder: the tensile stress developed by the string (it is equal to the given force **F**), the force of gravity mg, the force **N** exerted by the support on the cylinder (these two forces compensate each other), and the static-friction force $\mathbf{F}_{fr}$.

The static-friction force can assume any value within certain limits: $0 \ll F_{fr} \ll fN$. In the given case the fric-

tion force has a value that prevents slippage (pure rolling friction).

By the theorem on the motion of the center of mass,

$$ma_{CM} = F + F_{fr}.$$

The equation of motion about the axis passing through the center of mass yields

$$\frac{1}{2} mR^2\beta = (F - F_{fr}) R.$$

Allowing for the fact that

$$\beta = a_{CM}/R$$

and solving the obtained system of equations, we find that

$$a_{CM} = \frac{4F}{3m}, \quad F_{fr} = \frac{1}{3} F.$$

Substituting the numerical values, we get $a_{CM} = 4$ m/s^2 and $F_{fr} = 10$ N.

The condition of absence of slippage assumes the form

$$\frac{1}{3} F \ll fmg,$$

which yields

$$f > F/3\,mg \simeq 0.1.$$

If the friction coefficient is lower, slippage sets in.

Let us make the terms of the above problem more complicated. *Suppose that the string is swung over a massless pulley and a load of mass $m_2 = 20$ kg (a particle) is tied to the free end of the string; all other conditions remain unchanged with the exception of force* **F**, *which is now the tensile stress* $\mathbf{F}_t$ *developed by the string. Find the acceleration a_{CM} of the center of mass, the acceleration a of the load, and the tensile stress (Figure 14.5).*

We denote the cylinder's mass by m_1. Applying the dynamical method, we arrive at the following equations:

$m_2a = m_2g - F_t$ (for the translational motion of the load),

$m_1a_{CM} = F_t + F_{fr}$ (by the theorem on the center-of-mass motion),

$\frac{1}{2} mR^2\beta = (F_t - F_{fr}) R$ (the equation of motion).

The load's acceleration and the acceleration of the center of mass are connected by the relationship $a = a_{CM} + \beta R$. Since $a_{CM} = \beta R$, we have $a = 2a_{CM}$.

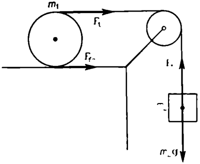

Figure 14.5

Solving this system of equations, we get

$$a=\left(1+\frac{3m_1}{8m_2}\right)^{-1} g, \quad a_{CM}=\frac{1}{2}\left(1+\frac{3m_1}{8m_2}\right)^{-1} g,$$

$$F_{fr}=\frac{m_1}{8}\left(1+\frac{3m_1}{8m_2}\right)^{-1} g, \quad F_t=\frac{3m_1}{8}\left(1+\frac{3m_1}{8m_2}\right)^{-1} g.$$

Substituting the numerical values yields

$$a \approx 8.4\ \mathrm{m/s^2}, \quad a_{CM} \approx 4.2\ \mathrm{m/s^2}, \quad F_{fr} \approx 10.5\ \mathrm{N},$$
$$F_t \approx 31.5\ \mathrm{N}.$$

The terms of the problem can be made still more complicated by allowing for the mass of the pulley, by assuming that the cylinder is a rigid body and moves along an inclined plane rather than a horizontal surface, by supposing that the load (particle) moves along an inclined plane, and the like. Clearly all these problems can be solved by applying the same dynamical method.

15. Conservation Laws in Rigid-Body Dynamics

The *elementary work* resulting from the rotation of a rigid body through an angle $d\varphi$ is defined by the following formula:

$$dW = M\, d\varphi, \qquad (15.1)$$

where M is the moment of force about the rotation axis. The total work is obtained by integrating Eq. (15.1):

$$W = \int_{\varphi_1}^{\varphi_2} M \, d\varphi. \tag{15.2}$$

The *kinetic energy* of a rigid body in arbitrary motion breaks down into the kinetic energy of translational motion and the kinetic energy of rotational motion:

$$E_k = mv_{CM}^2/2 + J\omega^2/2, \tag{15.3}$$

where v_{CM} is the velocity of translational motion of the center of mass, and J the moment of inertia of the rigid body about the rotation axis.

In addition to momentum and energy conservation laws in mechanics, rigid-body dynamics also employs the law of angular-momentum conservation. This law follows from the equation of motion (14.3) with respect to a point: *if the vector sum of the moments of external forces about a fixed point O is zero, the angular momentum about this point is constant*:

$$\mathbf{L} = \text{const.} \tag{15.4}$$

More often the law of angular-momentum conservation is used in a form that follows from the equation of motion (14.4) with respect to a fixed axis: *if the algebraic sum of the moments of external forces about a fixed axis is zero, the angular momentum of the system about this axis is a constant*:

$$L = \sum J\omega = \text{const}, \tag{15.5}$$

where the summation sign stands for the algebraic sum of the angular momenta of all the bodies in the system.

Application of conservation laws in rigid-body dynamics is carried out along the same lines as in particle dynamics.

Example 15.1. *A wooden rod of mass* $M = 6$ kg *and length* $l = 2$ m *can rotate in the vertical plane about a horizontal axis passing through point O (Figure 15.1). A bullet of mass* $m_0 = 10$ g, *flying with a velocity*

$v_0 = 10^3$ m/s *at right angles to the rod, hits the lower end and buries itself in the rod. Determine the kinetic energy of the rod after impact.*

Solution. Two bodies form the physical system: the rod and the bullet. The bullet can be thought of as a particle, while the rod is assumed to be a rigid body. The physical phenomenon consists of the bullet interacting with the rod (an inelastic collision). The state of the system prior to collision is known. We must determine a parameter of the system, the kinetic energy, after the interaction has ceased.

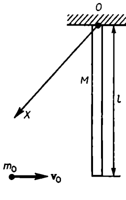

Figure 15.1

The nature of the forces acting in the interaction is assumed to be unknown. Therefore, the dynamical method cannot be applied. Let us apply the conservation-law method. Prior to collision, the bullet was moving rectilinearly, but after collision it is in rotational motion together with the rod. It is, therefore, advisable to employ the law of angular momentum conservation about the fixed rotation axis, since the conditions of applicability of this law are met.

As usual, the inertial reference frame is linked with Earth, the origin is placed at point O, the X axis is directed along the axis of rotation. The angular momentum of the bullet about the rotation axis prior to collision is $m_0 v_0 l$ and that of the rod is zero. After collision the angular momentum of the rod together with the bullet is $J\omega$, where J is the moment of inertia of rod and bullet about the X axis, and ω the angular velocity of their rotation after collision. Since the moment of inertia of the bullet is much smaller than that of the rod, $m_0 l^2 \ll (1/3)\ Ml^2$, we can assume that $J \approx (1/3)\ Ml^2$. By the law of conservation of angular momentum,

$$m_0 v_0 l = J\omega.$$

The kinetic energy of the rod is

$$E_k = \frac{J\omega^2}{2} = \frac{m_0^2 v_0^2 l^2}{2J} = \frac{3m_0^2 v_0^2}{2M}, \quad E_k = 25\ \text{J}. \quad (15.6)$$

Note that initially the kinetic energy of the bullet (prior to impact) was $E_{k0} = m_0 v_0^2/2$ that is, $E_{k0} = 5 \times 10^3$ J, which is considerably higher than the kinetic energy of the rod after impact. As a result of the inelastic collision the greater part of the initial mechanical energy was transferred into nonmechanical forms of energy. As a result of the interaction there emerged very strong nonconservative forces, which dissipated the mechanical energy of the system. Therefore, it would be wrong to use the law of conservation of energy in mechanics directly in the form $J\omega^2/2 = m_0 v_0^2/2$, and it would also be wrong to use the law of momentum conservation, since after collision the rod together with the bullet is in rotational motion. If we did use the law of momentum conservation, we would have $m_0 v_0 = (M + m_0)u$, where $u = \omega l$, whence, neglecting the bullet mass m_0 in comparison to the rod mass M, we would find that $\omega = m_0 v_0/Ml$. The kinetic energy of the rod after impact would be $J\omega^2/2 = m_0^2 v_0^2/6M$, which is roughly 2.7 J and lower than the result (15.6) almost by a factor of ten.

Let us find the maximal angle α by which the rod is deflected from the vertical after impact. There are no nonconservative forces in the system after impact and, hence, we can apply the energy conservation law to the motion of rod and bullet after collision. By this law,

$$\frac{3m_0^2 v_0^2}{2M} = Mgh,$$

where h is the height to which the rod's center of mass is raised (in relation to the point prior to impact, A) as a result of the impact (Figure 15.2). Here we have allowed for the fact that $m_0 \ll M$. From triangle OBC it follows that

$$\cos\alpha = \frac{l/2 - h}{l/2}.$$

Solving the system of equations, we get

$$\alpha = \cos^{-1}\left(1 - \frac{3m_0^2v_0^2}{M^2gl}\right), \quad \alpha \approx 54°.$$

We can also consider many variants of this problem, say, by replacing the bullet with a steel ball and the wooden rod with a steel rod, by assuming an elastic

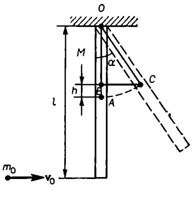

Figure 15.2

collision instead of an inelastic, by studying the case of a glancing collision, and the like. All these variants can be solved by the conservation-law method.

In conclusion we consider a problem whose solution will be found by all four methods: the kinematic, dynamical, conservation-law, and DI.

Example 15.2. *A solid homogeneous disk of radius* $R = 10$ cm *which initially had an angular velocity* $\omega_0 = 50$ rad/s *(about an axis perpendicular to the disk's plane and passing through the disk's center of mass) is placed on its base on a horizontal surface. How many rotations will the disk make before it stops if the coefficient of friction between the base and the horizontal surface,* f, *is* 10^{-1} *and does not depend on the disk's angular velocity?*

Solution. The physical system consists of a single body, the disk, which cannot be considered a particle (we assume

it to be a rigid body). The physical phenomenon consists of the disk being decelerated in its rotation about a fixed rotation axis under the forces of friction (all other forces are balanced). The initial and final states of the disk are known. We are looking for one of the parameters of this motion (the number of rotations N the disk will make before it stops). This constitutes a basic problem of rigid-body dynamics.

Let us apply the dynamical method. The disk's center of mass is at rest as the disk rotates. The equation of motion (14.7) yields

$$\frac{1}{2} mR^2\beta = M, \quad (15.7)$$

where $m = \pi R^2 h\rho$ is the disk's mass, h its height (thickness), ρ the density of its material, β the angular acceleration, and M the total moment of the forces of friction about the axis.

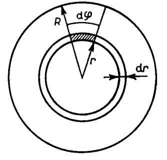

Figure 15.3

The force of friction is applied to each section of the disk, and since these sections lie at different distances from the axis, the moments of the forces of friction differ from section to section. To find M we apply the DI method. We partition the disk into thin rings (Figure 15.3). Each ring is also partitioned into small elements by neighboring radii that form a small angle $d\varphi$. In Figure 15.3 one such element is hatched. The force of friction acting on the element is

$$dF_{fr} = f\, d\varphi\, r\, dr\, h\rho g.$$

The moment of this force of friction is

$$dM = r\, dF_{fr} = f\rho g h r^2 dr\, d\varphi.$$

Integrating with respect to angle φ from zero to 2π and with respect to r from zero to R, we get the total mo-

ment of friction forces:

$$M = \int_0^{2\pi} \int_0^R f\rho g h r^2 \, dr \, d\varphi = f\rho g h 2\pi R^3/3 = 2fRgm/3. \qquad (15.8)$$

Substituting this value of M into the equation of motion (15.7), we find the angular acceleration of the disk:

$$\beta = 4fg/3R.$$

Solving the inverse problem of kinematics (the kinematic method), we determine the law of variation of the angular velocity,

$$\omega = \omega_0 - \beta t, \qquad (15.9)$$

and the respective law of motion,

$$\varphi = \omega_0 t - \beta t^2/2. \qquad (15.10)$$

Taking into account the fact that the final angular velocity of the disk is zero, $\omega = 0$, and employing Eq. (15.9), we can find the time of the rotation:

$$t = \frac{\omega_0}{\beta} = \frac{3R\omega_0}{4fg}, \quad t \approx 3.75 \text{ s}.$$

Substituting this value of t into Eq. (15.10) and bearing in mind that $\varphi = 2\pi N$, we get

$$N = \frac{3R\omega_0^2}{16\pi fg}, \quad N \approx 15. \qquad (15.11)$$

Now let us solve this problem using the conservation-law method. The physical system consists of two bodies: the disk and Earth. The system is closed, and the law of conservation of energy in mechanics could be employed if there were no nonconservative forces acting in the system. Assuming that these forces are external, we obtain from Eq. (13.16) the following:

$$J\omega_0^2/2 = W, \qquad (15.12)$$

where $J = (1/2)\, mR^2$ is the disk's moment of inertia, and W is the work performed by the nonconservative forces (friction). Since we already know the moment of these

forces (see (15.8)), we can use (15.2) to find

$$W=\int_0^{\varphi} M\,d\varphi=\int_0^{\varphi}\frac{2fRgm}{3}\,d\varphi=\frac{2fRgm\varphi}{3}.$$

Substituting this value of W into Eq. (15.12) and bearing in mind that $\varphi = 2\pi N$, we get

$$N=\frac{3R\omega_0^2}{16\pi fg},\quad N\approx 15,$$

which coincides with formula (15.11) found by using the dynamical method.

Concluding the study of the mechanical model, we see that any standard formulated problem in this department can be solved by applying a fairly small number of universal methods (aside from the method of analyzing the physical content of a problem): the kinematic, dynamical, conservation-law, and DI.

ELEMENTS OF THE THEORY OF PHYSICAL FIELDS

Chapter 5
THE GRAVITATIONAL FIELD

16. The Basic Problem of Gravitation Theory

The basic law of gravitation theory is Newton's law of gravitation,

$$F=G\frac{m_1m_2}{r^2},\tag{16.1}$$

where $G \approx 6.67 \times 10^{-11}$ N·m/kg^2 is the *universal gravitational constant*. In the form (16.1) the law is valid only for particles and spherical bodies. It can be written in vector form thus:

$$\mathbf{F}_{12}=-G\frac{m_1m_2}{r^3}\mathbf{r},\tag{16.2}$$

where $\mathbf{F}_{12}$ is the vector of the gravitational force acting on body m_2, and $\mathbf{r}$ the radius vector pointing from body m_1 to body m_2 (Figure 16.1).

The main characteristic of each point in a gravitational field is the *field strength* **E**, a vector quantity defined thus:

$$\mathbf{E} = \mathbf{F}/m_0, \qquad (16.3)$$

where **F** is the gravitational force acting on a particle of mass m_0 placed at the given point.

The strength and *potential* of the gravitational field generated by a particle of mass m at a point positioned

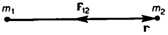

Figure 16.1

at a distance r from the mass are expressed by the following formulas:

$$E = Gm/r^2, \qquad (16.4)$$

$$\varphi = -Gm/r. \qquad (16.5)$$

The field strength **E** at a point and the potential φ at the same point in a gravitational field are linked by the formula

$$\mathbf{E} = -\text{grad}\ \varphi. \qquad (16.6)$$

The state of the physical system considered (the gravitational field) is determined by the value and direction of vector **E** at any point in the field. The field strength **E** of a gravitational field is the fundamental characteristic in the sense that, knowing **E**, we can not only determine any parameter characterizing the field but describe the behavior of physical systems in this field. Indeed, Eq. (16.6) can be used to find potential φ, and Eq. (16.3) makes it possible to determine the force with which the field acts on a body placed in it. If the initial conditions for this body are known, by applying the dynamical method we can determine the law of the body's motion. And knowing this law makes it possible to find all other characteristics and parameters determining the motion. This leads to the following formulation of the basic problem in gravitation theory, the problem of calculating the field.

Calculating a gravitational field means determining at each point in the field the field strength vector **E** or the potential φ.

17. The Gravitational Field Generated by a System of Particles

A fundamental physical principle lies at the base of the method for calculating physical fields, the *superposition principle*. If the field is generated by a system of particles, we must first determine the field generated by each particle separately (i.e. the corresponding vector $\mathbf{E}_i$). Then in accordance with the superposition principle we find the resultant field vector **E** as the vector sum of the field strength vectors:

$$\mathbf{E} = \mathbf{E}_1 + \ldots + \mathbf{E}_i + \ldots + \mathbf{E}_n. \quad (17.1)$$

The gravitational field generated by a single particle was calculated in Section 16. Description of the motion of even one body in the gravitational field generated by one particle presents certain mathematical difficulties. It is fairly easy to solve such a problem physically, that is, set up a closed system of equations, by employing either the dynamical method or the conservation-law method. Difficulties for first-course students appear at the mathematical level, when it is necessary to solve the system of equations (often differential equations).

First it is advisable to solve a number of elementary problems that involves estimation, say calculate the field strength and potential of the gravitational field on the surface of the Moon, the Sun, or Mars (note that $g = GM/R^2 \approx 9.8$ m/s^2 is the gravitational field strength at Earth's surface), determine (estimate) the orbital and escape velocities for Earth, the Moon, and Mars, and the like.

Then the first problem concerning the motion of a particle in a known gravitational field can be formulated. It is even advisable to present it in the form of a nonspecified problem.

Example 17.1. *A rocket is launched vertically with an initial velocity v_0 at the North Pole (it is assumed that the*

boosters impart a velocity of v_0 to the rocket instantaneously and are then switched off). Describe the motion of the rocket.

Solution. The problem is nonspecified. The first simplifying assumption is obvious: the air drag is ignored. The rocket can be thought of as being a particle. Its motion can be described if we find its law of motion. The law of motion depends essentially on the value of the initial velocity v_0. Let us assume that v_0 is so low that at the point of the rocket's greatest altitude the acceleration of free fall g_1 (which constitutes the strength of Earth's gravitational field) differs very little (say, not more than by one percent) from the acceleration of free fall on Earth's surface, g_0. It is useful to estimate this altitude h_1 and the initial velocity v_{01} that the rocket must have to reach this altitude. Since, by assumption, $(g_0 - g_1)/g_0 = 10^{-2}$ and

$$g_0 = \frac{GM}{R^2}, \quad g_1 = \frac{GM}{(R+h_1)^2},$$

we have $h_1 \approx R\ (1/\sqrt{0.99} - 1)$ and $v_{01} = \sqrt{2g_0h_1}$, that is, $h_1 \approx 32$ km and $v_{01} \approx 800$ m/s.

Thus, if $v_0 \ll v_{01}$, then the rocket's acceleration is approximately constant, and we have a trivial high-school problem concerning the uniformly decelerated upward motion of a particle with a constant acceleration g_0. The law of motion in this case can be written in the form

$$x = v_0t - g_0t^2/2,$$

which provides all the parameters of motion.

We will not consider the case where the initial velocity v_0 is greater than or equal to $v_2 \approx 11.2$ km/s, the escape velocity for Earth. Thus, the first problem can be formulated as follows:

Example 17.2. *A rocket is launched vertically at Earth's North Pole with an initial velocity v_0 satisfying the conditions $v_{01} < v_0 < v_2$. Find the law of its motion. Ignore air drag and the effect of other planets, the Sun, and the Moon on the motion of the rocket.*

Solution. Two bodies constitute the physical system, the rocket and Earth. The rocket can be thought of as

being a particle. The gravitational field generated by Earth (assumed to be a spherical body) is known. The physical phenomenon consists of the particle (rocket) moving in a nonuniform gravitational field. We wish to determine the law of motion of the rocket. This constitutes the basic problem of particle dynamics.

We apply Newton's second law. The inertial reference frame is that linked with Earth (since Earth's mass is considerably greater than the rocket's mass, we assume Earth to be fixed), the X axis is directed upward, and the origin is placed at Earth's center. There is only one force acting on the rocket, the gravitational. It is important to note that this force is variable. Then, by Newton's second law,

$$m\ddot{x} = -GmM/x^2 \quad \text{for} \quad x \gg R. \qquad (17.2)$$

Physically the problem has been solved because we have obtained a single differential equation in one unknown function $x(t)$, the coordinate of the rocket in space. Solving the equation we get the law of motion. However, first-year students have great difficulty in solving this equation. Two points must be stressed in this connection. First, Eq. (17.2) is solvable in principle, and in the final analysis we can obtain the law of the motion of the rocket. Second, even at this level students can be told that solving physics problems may lead to equations that have no exact (analytical) solutions. Then a computer must be brought into the picture to obtain numerical, approximate solutions.

Let us simplify the formulation of the problem by using the conservation-law method instead of the dynamical. We apply the law of energy conservation to the rocket-Earth system:

$$\frac{mv_0^2}{2} - \frac{mGM}{R} = \frac{mv^2}{2} - \frac{mGM}{x}, \qquad (17.3)$$

where v is the velocity of the rocket at a point with coordinate x. This gives us the maximum value of x (the position of the rocket at $v = 0$):

$$x_{\max} = \frac{2GMR}{2GM - v_0^2 R} \qquad (17.4)$$

If the initial velocity v_0 is equal, say, to the orbital velocity for Earth, $v_1 = \sqrt{GM/R}$, the maximum value of x is $x_{max} = 2R$, and the maximum altitude reached by the rocket is $h_{max} = R \approx 6400$ km. Equation (17.3) can be used to find the dependence of the rocket's velocity on coordinate x:

$$v = \sqrt{v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{x}\right)}. \qquad (17.5)$$

The diagram of this dependence is shown in Figure 17.1. Now we can formulate the second (simpler) problem.

Example 17.3. *A rocket is launched vertically at Earth's North Pole with an initial velocity v_0 satisfying the conditions $v_{01} < v_0 < v_2$. Determine the maximum altitude*

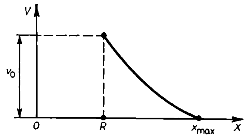

Figure 17.1

reached by the rocket and the rocket's velocity at an arbitrary point of its path. Ignore air drag and the effect of other planets, the Sun, and the Moon on the motion of the rocket.

Solution of this problem was obtained earlier (see Eqs. (17.4) and (17.5)).

Note that in the above problems we must appraise in more detail the upper bound on the initial velocity, since at velocities close to v_2 the altitude reached by the rocket is so great that the effect of the Moon, Sun, and other bodies can no more be ignored. It is advisable for more advanced students to make the necessary estimates.

To conclude this section let us consider one more problem.

Example 17.4. *A rocket is circling Earth along an orbit that almost coincides with the Moon's orbit. When a retroengine is fired, the rocket rapidly loses speed and begins to fall toward Earth (Figure 17.2). Determine the time it will take the rocket to reach Earth. Ignore air drag and the effect of other bodies on the rocket's motion.*

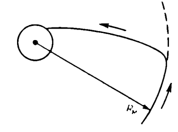

Figure 17.2

Solution. The physical system consists of the rocket and Earth. The physical phenomenon consists of the rocket moving in Earth's gravitational field. The solutions of the previous problems in this section show that the dynamical method leads to a complicated differential equation, while the conservation-law method makes it possible to find the rocket's velocity at each point of the path but not the sought time of fall. So standard methods have led us nowhere. But suppose that we consider the motion of the rocket as the motion of a satellite of Earth along an extremely prolate ellipse whose major axis is as long as the radius of the Moon's orbit, $R_M \approx 4 \times 10^5$ km and whose eccentricity $\varepsilon = 1$. This means that we can employ Kepler's third law,

$$\left(\frac{2t}{T}\right)^2 = \left(\frac{(1/2)\,R_M}{R_M}\right)^3,$$

where t is the time of fall, and $T = 27.3$ days is the period of revolution of the Moon about Earth. Calculations yield $t = T/4\sqrt{2}$, which amounts to about 4.85 days.

18. The Gravitational Field Generated by an Arbitrary Mass Distribution

The common approach in calculating the gravitational field generated by an arbitrary mass distribution is to employ the superposition principle and the DI method (see Section 6). In calculating the field strength by this

method it is extremely important to take into account the vector nature of this quantity. After the elementary field strength vector dE has been found, its projections dE_x, dE_y, and dE_z on the respective coordinate axes are determined, with subsequent integration (summation) carried out for each projection separately.

If the field strength is known, the problem on the motion of bodies in such fields is solved by either the dynamical method or the conservation-law method.

Example 18.1. *Describe the motion of a particle in the gravitational field generated by a long thin, homogeneous rod of mass M and length l. The effect of other bodies can be ignored.*

Solution. Let us restrict our discussion to the one-dimensional case. We assume that initially the particle

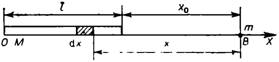

Figure 18.1

was positioned on the rod at a distance $x_0 = l$ from one of the rod's ends (point B in Figure 18.1) and had a zero velocity ($v_0 = 0$). The physical system consists of two bodies: the rod and the particle (whose mass we denote by m). The physical phenomenon consists of the particle moving in the gravitational field generated by the rod.

The gravitational force acting on the particle is unknown (it is not $F = GmM/x^2$ since the rod is not a particle). To use the dynamical method we must calculate the gravitational field generated by the rod on the rod's axis, that is, we must find the field strength vector E and potential φ. Let us apply the DI method. We assume that $m \ll M$. We select the inertial reference frame as the one linked with the rod, place the origin O at the left end of the rod, and direct the X axis to the right. We partition the rod into segments so small that each

can be thought of as being a particle. We take one segment of length dx positioned at a distance x from an arbitrary point B on the axis. The mass of the segment is $dm = \rho A\, dx$, where A is the cross-sectional area of the rod, and ρ the density of the rod's material. Since the selected segment is a particle, the characteristics of the gravitational field generated by it (the field strength dE and potential $d\varphi$) are known:

$$dE = \frac{G\, dm}{x^2} = \frac{G\rho A\, dx}{x^2}, \quad d\varphi = -\frac{G\, dm}{x} = -\frac{G\rho A\, dx}{x}.$$

Note that in our case all the elementary field strength vectors $d\mathbf{E}$ point in the same direction. Integration yields the overall characteristics of the field generated by all the elementary segments (i.e. the field generated by the rod as a whole):

$$E = \int_{x_0}^{l+x_0} \frac{G\rho A\, dx}{x^2} = \frac{GM}{x_0(l+x_0)},$$

$$\varphi = -\int_{x_0}^{l+x_0} \frac{G\rho A\, dx}{x} = -\frac{GM}{l} \ln\left(1+\frac{l}{x_0}\right).$$

The force acting on a particle placed at a distance x from the origin is

$$\mathbf{F} = -\frac{GMm}{x(x+1)}\,\mathbf{i}.$$

Newton's second law,

$$-\frac{d^2x}{dt^2} = -\frac{GM}{x(x+l)},$$

results in a differential equation whose solution will enable us to establish the appropriate law of motion of the particle.

Applying the law of energy conservation in mechanics,

$$-\frac{GMm}{l} \ln\left(1+\frac{l}{x_0}\right) = -\frac{GMm}{l} \ln\left(1+\frac{l}{x}\right) + \frac{mv^2}{2},$$

we can determine the velocity of the particle positioned at a distance x from the right end of the rod:

$$v=\sqrt{\frac{2GM}{l}\ln\frac{1+l/x}{1+l/x_0}}.$$

Let us consider examples of more complicated fields.

Example 18.2. *Determine the field strength of the gravitational field generated by a thin ring of radius R and*

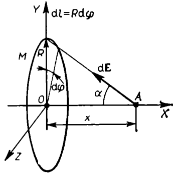

Figure 18.2

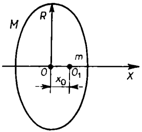

Figure 18.3

mass M at a point A lying on the ring's axis at a distance x from the ring's plane (Figure 18.2).

Solution. The physical system consists of the ring and the ring's gravitational field. We wish to solve a basic problem of gravitational theory, namely, find the field generated by the ring. A ring, however, cannot be thought of as a particle and, hence, formula (16.4) is invalid.

Let us employ the DI method. We partition the ring into segments so small that each segment can be regarded as a particle. We take one such segment of length $dl = R\,d\varphi$ (see Figure 18.2). This generates a gravitational field whose field strength vector at point A is $d\mathbf{E}$ of a magnitude

$$dE=\frac{G\,dM}{R^2+x^2}, \tag{18.1}$$

where $dM = (M/2\pi R)\,dl$ is the mass of element dl. Vector $d\mathbf{E}$ forms an angle α with the X axis and an angle φ with the Z axis. The projection of $d\mathbf{E}$ on the X axis is

$$dE_x = \frac{G\,dM}{R^2+x^2}\cos\alpha = \frac{GMx\,d\varphi}{2\pi(R^2+x^2)^{3/2}}. \qquad (18.2)$$

Integration of (18.2) with respect to φ yields the projection of the sought field vector on the X axis:

$$E_x = \int_0^{2\pi} \frac{GMx\,d\varphi}{2\pi(R^2+x^2)^{3/2}} = \frac{GMx}{2\pi(R^2+x^2)^{3/2}}\int_0^{2\pi} d\varphi = \frac{GMx}{(R^2+x^2)^{3/2}}. \qquad (18.3)$$

We can clearly see that in view of the symmetry of the problem the sum of the projections of the elementary field strengths on the Y axis is zero: $E_y = 0$; so is the sum of the projections on the Z axis: $E_z = 0$. Hence, the sought field vector is directed along the X axis and its magnitude is given by (18.3). After calculating the field generated by a ring we can formulate a number of problems on the motion of bodies in such a field.

Example 18.3. *Describe the motion of a particle of mass m that initially was at rest at point O_1 on the axis of a thin ring of mass M and radius R; point O_1 lies at a distance $x_0 \ll R$ from the ring's plane. Assume that $m \ll M$ (Figure 18.3).*

Solution. To describe the motion of a particle in a known gravitational field means to find the law of motion of this particle. This constitute a basic problem of dynamics.

Using the dynamical method, from Newton's second law we obtain the differential equation of harmonic oscillations:

$$m\ddot{x} = -\frac{GMm}{R^3}x. \qquad (18.4)$$

Here we have allowed for the fact that for $x \ll R$ the strength of the gravitational field generated by the ring is given by (18.3), or $E \approx (GM/R^3)\,x$. Thus, the particle oscillates harmonically according to the law of mo-

tion $x = x_0 \sin(\omega_0 t + \alpha_0)$, the period of these oscilla tions being $T = 2\pi R\sqrt{R/GM}$.

If the condition $x_0 \ll R$ is not met, the same dynamical method leads to a more complicated differential equation:

$$m\ddot{x} = -\frac{GMm}{(R^2+x^2)^{3/2}}x. \tag{18.5}$$

Solution of the equation will result in finding the law of motion $x = x(t)$, and using the law of energy conservation, we can find the dependence of the velocity v of the particle on the position (coordinate x) of the particle, but first we must determine the potential of the gravitational field generated by the ring by applying either the DI method or formula (16.6).

Let us make the terms of Example 18.2 more complicated. *We wish to calculate the strength of the gravitational field generated by a semiring at the same point.* Clearly, the projection of the resulting field strength vector on the X axis diminishes by a factor of two in this case:

$$E_x = \frac{GMx}{2(R^2+x^2)^{3/2}}, \tag{18.6}$$

But the most important fact is that because the symmetry of the problem breaks down, there emerges a nonzero projection of the field strength vector on the Z axis, E_z. Since

$$\mathrm{d}E_z = \frac{GM \sin\alpha \cos\varphi \, dl}{2\pi R(R^2+x^2)} = \frac{GMR\cos\alpha \, d\varphi}{2\pi(R^2+x^2)^{3/2}}, \tag{18.7}$$

we have

$$E_z = \int_{-\pi/2}^{+\pi/2} \frac{GMR\cos\varphi \, d\varphi}{2\pi(R^2+x^2)^{3/2}} = \frac{GMR}{\pi(R^2+x^2)^{3/2}}. \tag{18.8}$$

Note that in view of the remaining symmetry $E_y = 0$.

The terms of Example 18.2 could be made still more complicated, say, if the aim were to *calculate the gravitational field generated by a quarter of a ring, an arc with a central angle $\varphi < \pi/2$*, and so on. All these problems can be solved by the same method.

Let us employ the above results to calculate the field generated by a hemisphere.

Example 18.4. *A sphere of mass M and radius R is separated into two hemispheres by a plane passing through its center. Determine the potential of the gravitational field generated by each hemisphere at a point O on a straight line*

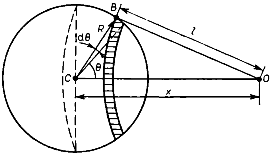

Figure 18.4

that is perpendicular to the separating plane and passes through the sphere's center; the distance from point O to the center of the sphere is $x > R$ (Figure 18.4).

Solution. To calculate the gravitational field generated by each hemisphere (not a particle) we employ the DI method. We partition each hemisphere into narrow rings of width $R\,d\theta$ and radius $R \sin\theta$. Now we consider one ring. Its surface area is

$$dA = 2\pi R \sin\theta\, R\, d\theta = 2\pi R^2 \sin\theta\, d\theta \qquad (18.9)$$

and its mass is

$$dM = \frac{dA}{4\pi R^2} M = \frac{M \sin\theta\, d\theta}{2}. \qquad (18.10)$$

Since all the elements of this ring are positioned at the same distance l from point O, the elementary potential of the gravitational field generated by the ring at point O is

$$du = -G\frac{dM}{l} = -\frac{GM}{2}\frac{\sin\theta\, d\theta}{l}. \qquad (18.11)$$

Take triangle CBO. By the law of cosines,

$$l^2 = R^2 + x^2 - 2Rx\cos\theta. \tag{18.12}$$

Differentiation of both sides of (18.12) yields

$$2l\,dl = 2Rx\sin\theta\,d\theta,$$

which implies that

$$\sin\theta\,d\theta = l\,dl/Rx.$$

Substituting this into (18.11), we arrive at the final expression for the elementary potential du as a function of a single variable, l:

$$du = -\frac{G\,dM\,dl}{2Rx}. \tag{18.13}$$

The elementary potential could also be expressed as a function of a single variable θ by excluding l from (18.11) via (18.12). But the subsequent integration in this case would be more complicated than integration of (18.13).

Denoting the potential generated by the right hemisphere by U_1 and that generated by the left by U_2 and integrating (18.13), we get

$$U_1 = \int_{x-R}^{\sqrt{R^2+x^2}} \left(-\frac{GM}{2Rx}\right) dl = -\frac{GM}{2Rx}\,[\sqrt{R^2+x^2} - (x-R)], \tag{18.14}$$

$$U_2 = \int_{\sqrt{R^2+x^2}}^{x+R} \left(-\frac{GM}{2Rx}\right) dl = -\frac{GM}{2Rx}\,[(x+R) - \sqrt{R^2+x^2}]. \tag{18.15}$$

Comparing (18.14) with (18.15), we arrive at an expression for the potential of the gravitational field generated by a sphere at a point lying outside the sphere:

$$U = U_1 + U_2 = -GM/x. \tag{18.16}$$

The same method can be used to consider the case where $x < R$. We can also calculate the gravitational field generated outside and inside a homogeneous ball, which was partially done in the problem of oscillations of an

object in a shaft (e.g. see Example 12.1)), etc. To conclude this section let us consider the following problem.

Example 18.5. *A spacecraft has positioned itself at a point on the axis of a planetary nebula. The point is at a distance* $r_0 = 5d$ *from the nebula's center of mass, and the rocket engines of the spacecraft have been switched off. How much time will it take the spacecraft to reach the nebula, moving only because of gravitational attraction? The nebula is assumed to be a disk of diameter* $d = 10^{-2}$ parsec, *thickness (depth)* $h = 10^{-3}$ parsec, *and homogeneous distribution of matter with a density* $\rho = 10^{-17}$ kg/m^3. *The initial velocity of the spacecraft with respect to the nebula is assumed zero,* $v_0 = 0$, *and the spacecraft mass is* $m = 10^5$ kg (1 parsec $= 3.08 \times 10^{13}$ km $= 3.08 \times 10^{16}$ m).

Solution. The physical system consists of two physical objects, the nebula and the spacecraft. The latter can be considered a particle. Since the depth of the nebula is small compared to both the distance r_0, and the diameter d of the nebula, we will assume that the nebula is a thin disk. The physical phenomenon consists of a particle (spacecraft) moving in the gravitational field generated by the nebula (the disk is not a particle). We wish to find the time of flight of the spacecraft. The law of motion can be determined by the dynamical method if we know the strength of the gravitational field generated by the nebula.

Thus, the solution procedure is clear: we must first calculate the gravitational field generated by a thin disk (nebula) of mass

$$M_n = \pi d^2 h\rho/4, \qquad (18.17)$$

then via the dynamical method determine the law of motion of the spacecraft, and then find the time of flight from the law of motion.

Let us calculate the strength of the gravitational field generated by the disk. We employ the DI method. We partition the disk into thin rings of width dr and consider one ring of radius r (Figure 18.5). Its mass is

$$dM = 2\pi r h\rho \, dr. \qquad (18.18)$$

Using formula (18.3), we can find the elementary strength of the gravitational field generated by the ring:

$$dE_n = \frac{2\pi G r_0 h\rho r\, dr}{(r^2+r_0^2)^{3/2}}. \tag{18.19}$$

Integration of (18.19) yields the strength of the gravita-

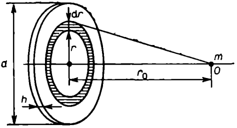

Figure 18.5

tional field generated by the entire nebula:

$$E_n = \int_0^{d/2} \frac{2\pi G r_0 \rho h r\, dr}{(r^2+r_0^2)^{3/2}} = 2\pi G\rho h\left(1 - \frac{1}{\sqrt{1+(d/2r_0)^2}}\right). \tag{18.20}$$

Next, applying the dynamical method, we find that Newton's second law leads us to the following differential equation:

$$m\ddot{x} = -2\pi G\rho h m\left(1 - \frac{1}{\sqrt{1+(d/2x)^2}}\right). \tag{18.21}$$

Solution of this equation would enable us to find the law of motion of the spacecraft, $x = x(t)$, and, hence, calculate the time of flight t_0. But it seems that there was no need to set up the differential equation (18.21) and even less need to solve the equation. The sought time of flight t_0 can be found approximately in a simpler way, namely, by the estimate method. We start by estimating the mass of the nebula. Formula (18.17) yields

$$M_n = \pi d^2 h\rho/2, \quad M_n \approx 2\times 10^{25} \text{ kg}.$$

Thus, the nebula's mass is very small: it is less than the solar mass $M_S = 2 \times 10^{30}$ kg by a factor of 10^6. Allowing for the fact that dimensions of the nebula are great (the depth of the nebula $h = 3 \times 10^{13}$ m is several times greater than the diameter of the solar system $d_0 \approx 1.2 \times 10^{13}$ m), we can easily predict that the gravitational field generated by the nebula will be very weak even at the nebula's boundaries.

Let us estimate the order of magnitude of E_n for two values of r_0: (1) $r_0 = 5d$, and (2) $r_0 \approx 0$ (at the boundary of the nebula). From (18.20) we find that $E'_n \approx 10^{-15}$ m/s^2 and $E''_n \approx 10^{-13}$ m/s^2, respectively. These values are extremely small. Even if the spacecraft is moving with maximum acceleration $a = E''_n$, it will take it

$$t_1 = \sqrt{2s/E''_n}, \quad t_1 \approx 4.5 \times 10^6 \text{ s} \approx 52 \text{ days},$$

to cover the distance $s = 1$m, and

$$t_0 = \sqrt{5d/E''_n},$$

which amounts to about 1.7×10^{14} s, or 5×10^6 years, to cover the distance $r_0 = 5d$.

Thus, in such a weak gravitational field the spacecraft is practically at rest. This is the result of applying the estimate method for solving the problem. The result is instructive and demonstrates that sometimes, before applying the laws of physics and setting up (differential) equations, it is advisable to make a rough estimate of several quantities and analyze (compare) the results obtained in the process.

Chapter 6
THE ELECTRIC FIELD

19. The Electrostatic Field in a Vacuum

The basic law of electrostatic-field theory is Coulomb's law,

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}. \tag{19.1}$$

The law is valid for point electric charges that are at rest. It closely resembles Newton's law of gravitation. Hence, everything said in Chapter 5 concerning a gravitational field can be said of an electrostatic field.

The main characteristics of an electrostatic field are the *field strength* **E** and the *potential* φ. For a field generated by a point charge we have

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \tag{19.2}$$

$$\varphi = \frac{Q}{4\pi\varepsilon_0 r}. \tag{19.3}$$

The field strength **E** and potential φ of an electrostatic field are linked by formula (16.6).

The state of an electrostatic field as a physical system is determined entirely by the direction and magnitude of the field strength vector at every point in the field. Hence, the basic problem of electrostatics consists of calculating the electric field. Here it is advisable to distinguish three cases:

(1) the field is generated by a system of point charges;

(2) the field is generated by a system of point charges and charges carried by bodies of regular shape; and

(3) the field is generated by an arbitrary distribution of electric charge.

Although the first case was considered earlier in connection with the gravitational field, it is highly advisable to calculate the fields generated by a dipole (not only at points lying on the dipole's axis but at arbitrary points), a quadrupole, and other point-like systems. In the second case we first use Gauss' law of flux to calculate the fields generated by charges distributed over regularly shaped objects and then, using the superposition principle, determine the total field. For an arbitrary distribution of charge we employ the DI method (see Section 6).

If the characteristics of the field have been calculated, problems on the motion of electrically charged particles in a known field can be solved by either the dynamical method or the conservation-law method.

Example 19.1. *Calculate the strength of an electric field generated by a straight infinitely long string uniformly*

charged with a linear density γ, *at a point O that is* r_0 *distant from the string.*

Solution. Since the charge cannot be considered point-like, we cannot use formula (19.2). Let us apply Gauss' law of flux. In view of the symmetry of the field, the field

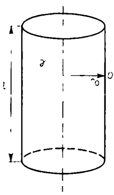

Figure 19.1

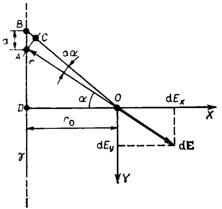

Figure 19.2

strength vector at each point is directed along the normal to the cylindrical surface on which the point lies, with the symmetry axis of this cylindrical surface coinciding with the string. For this reason, for the closed surface we take a cylinder of length l whose symmetry axis coincides with the string and on whose lateral surface point O lies (Figure 19.1). The flux of vector **E** passing out of the lateral surface of the cylinder is $\Phi_E = 2\pi r_0 lE$, and the electric charge inside the cylinder is $Q = \gamma l$. By Gauss' law,

$$2\pi r_0 lE = \gamma l/\varepsilon_0.$$

Hence, the sought field strength is

$$E = \frac{\gamma}{2\pi\varepsilon_0 r_0}. \tag{19.4}$$

Now let us approach this problem from the DI angle. We partition the string into segments so small that the

charge carried by each can be considered point-like. We select one such segment of length dl carrying a charge $dQ = \gamma\, dl$ (Figure 19.2). At point O the elementary field generated by this charge has a strength of

$$dE = \frac{dQ}{4\pi\varepsilon_0 r^2} = \frac{\gamma\, dl}{4\pi\varepsilon_0 r^2}. \qquad (19.5)$$

From triangle ADO we get

$$r = r_0/\cos\alpha.$$

Since $|AC| = r\, d\alpha = r_0\, d\alpha/\cos\alpha$, we find that triangle ABC yields

$$dl = |AC|/\cos\alpha = r_0\, d\alpha/\cos^2\alpha.$$

Substituting the values of r and dl into Eq. (19.5), we get

$$dE = \frac{\gamma\, d\alpha}{4\pi\varepsilon_0 r_0}. \qquad (19.6)$$

The projections of vector $\mathbf{dE}$ on the X and Y axes are

$$dE_x = \frac{\gamma\cos\alpha\, d\alpha}{4\pi\varepsilon_0 r_0}, \qquad (19.7)$$

$$dE_y = \frac{\gamma\sin\alpha\, d\alpha}{4\pi\varepsilon_0 r_0}. \qquad (19.8)$$

Integration yields

$$E_x = \int_{-\pi/2}^{+\pi/2} \frac{\gamma\cos\alpha\, d\alpha}{4\pi\varepsilon_0 r_0} = \frac{\gamma}{2\pi\varepsilon_0 r_0}, \quad E_y = \int_{-\pi/2}^{+\pi/2} \frac{\gamma\sin\alpha\, d\alpha}{4\pi\varepsilon_0 r_0} = 0.$$

Thus, the final result is

$$E = \frac{\gamma}{2\pi\varepsilon_0 r_0},$$

which coincides with formula (19.4) obtained via Gauss' law.

At first glance the DI method seems to have proved more involved than the Gauss'-law approach. In the given example this is indeed the case. But the DI method is universal and can be applied in cases where the Gauss'-law approach proves useless.

Example 19.2. *Determine the strength of the electric field generated by a straight piece of string carrying an elec-*

tric charge with a linear density γ, *at a point O that is* r_0 *distant from the string. The angles* α_1 *and* α_2 *are specified (Figure 19.3).*

Solution. Clearly, the symmetry of the field generated by an infinite string is broken: the field is not symmetric. It is extremely difficult to enclose the piece of string

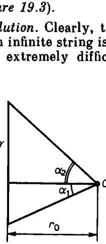

Figure 19.3

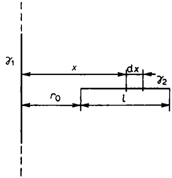

Figure 19.4

with a surface using which it would be fairly easy to calculate, via Gauss' law, the flux of vector **E**.

Let us apply the DI method. The projections of the elementary field vector d**E** on the X and Y axes were obtained in Example 19.1. Integrating (19.7) and (19.8), we find the projections (or components) of the sought vector **E** on the X and Y axes:

$$E_x = \int_{-\alpha_1}^{+\alpha_2} \frac{\gamma \cos \alpha \, d\alpha}{4\pi\varepsilon_0 r_0} = \frac{\gamma}{4\pi\varepsilon_0 r_0} (\sin \alpha_1 + \sin \alpha_2), \quad (19.9)$$

$$E_y = \int_{-\alpha_1}^{+\alpha_2} \frac{\gamma \sin \alpha \, d\alpha}{4\pi\varepsilon_0 r_0} = \frac{\gamma}{4\pi\varepsilon_0 r_0} (\cos \alpha_1 - \cos \alpha_2). \quad (19.10)$$

Clearly, the field generated by a charged infinitely straight string, (19.4), constitutes a particular case of the field generated by a piece of charged straight string. In-

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deed, for $\alpha_1 = -\pi/2$ and $\alpha_2 = +\pi/2$, Eqs. (19.9) and (19.10) yield $E_x = \gamma/2\pi\varepsilon_0 r_0$ and $E_y = 0$, which coincide with (19.4).

Now that we have formulas for expressing the strength of fields generated by a segment and an infinitely long charged string, we can formulate dozens of problems involving the calculation of fields generated by various combinations of uniformly charged segments, and infinite and semi-infinite strings (triangles, squares, angles, etc.).

Example 19.3. *An infinitely long string uniformly charged with a linear density* $\gamma_1 = +3 \times 10^{-7}$ C/m *and a segment of length* $l = 20$ cm *uniformly charged with a linear density* $\gamma_2 = +2 \times 10^{-7}$ C/m *lie in a plane at right angles to each other and separated by a distance* $r_0 = 10$ cm *(Figure 19.4.) Determine the force with which these two bodies interact.*

Solution. Two objects constitute the physical system, the infinitely long string and the segment. Neither of the two can be considered a particle. The physical phenomenon consists of the effect that the field of the string has on the charge of the segment. We wish to find the force of this interaction. The charge $Q_2 = \gamma_2 l$ carried by the segment is positioned in the electric field of the string, which is known (see (19.4)).

It would seem that to find the force acting on the charge we need only use the formula $F = Q_2 E$, where $E = \gamma_1/2\pi\varepsilon_0 r_0$. This is not correct, however, since the formula is valid either in the case of a homogeneous electric field (the electric field generated by the string is nonhomogeneous: $\mathbf{E} \neq$ const) or in the case of a point charge (Q_2 is distributed over the segment). On different sections (of equal length) of the segment of length l different forces are acting. Therefore, to calculate the force with which the nonhomogeneous field generated by the string acts on the distributed charge Q_2 we apply the DI method. We partition segment l into sections of length dx so small that the charge $dQ = \gamma_2\, dx$ of each section can be considered a point charge. The charge dQ is in the electric field of the string. Since this is a point charge, the force

acting on it is

$$dF = E\,dQ = \frac{\gamma_1\gamma_2}{2\pi\varepsilon_0 x}\,dx, \qquad (19.11)$$

where x is the distance from charge dQ to the string.

We now have the differential of the sought quantity. The force acting on each section of the segment depends on the distance x from the segment to the string, and so we select x as the variable of integration (it varies from $x_1 = r_0$ to $x_2 = r_0 + l$. Integrating Eq. (19.11) with respect to x, we get

$$F = \int_{r_0}^{r_0+l} \frac{\gamma_1\gamma_2}{2\pi\varepsilon_0}\frac{dx}{x} = \frac{\gamma_1\gamma_2}{2\pi\varepsilon_0}\ln\left(1+\frac{l}{r_0}\right).$$

Substitution of numerical values yields the result $F \approx 1.2 \times 10^{-3}$ N.

The terms of the above problem can be changed by placing the segment parallel to the string, at an angle to the string, in a plane perpendicular to the string, and so on. All these variants can be solved by the same DI method.

Let us consider fields generated by curved charged lines, surfaces, etc.

Example 19.4. *A piece of string is laid out in the form of a semicircle of radius $R = 2$ m and is charged uniformly with an electric charge $Q = 10^{-9}$ C. Find the strength of the electric field generated by the string at the point that is the geometrical center of the semicircle.*

Solution. The physical system consists of two objects, the semicircle uniformly charged with Q and the electric field generated by the charge. The field strength is unknown. The charge carried by the semicircle cannot be a point charge since the size of the semicircle, πR, is comparable to the distance R considered in the problem, so that the solution

$$E = \frac{Q}{4\pi\varepsilon_0 R^2}, \quad E \approx 2.2 \text{ V/m}, \qquad (19.12)$$

is wrong. Gauss' law involves extremely complicated calculations. We, therefore, turn to the DI method.

Let us select the inertial reference frame as the one linked to the semicircle and direct the X axis as shown in

Figure 19.5. We partition the semicircle into arcs of length dl so small that the charge $dQ = Q\, dl/\pi R$ carried by an arc can be considered a point charge. We take one such point charge. It generates an electric field whose strength vector $d\mathbf{E}_1$ at a point A makes an angle α with the X axis. Obviously, to each elementary charge in the upper half-plane there corresponds a symmetrically placed charge in the lower half-plane. The vector sum of $d\mathbf{E}_1$

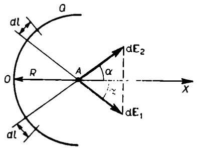

Figure 19.5

and $d\mathbf{E}_2$ is a vector directed along the X axis. Hence, in the summation process we need only take into account the projections of the elementary field strength vectors on the X axis:

$$dE_x = dE_1 \cos\alpha = \frac{dQ}{4\pi\varepsilon_0 R^2} \cos\alpha = \frac{Q \cos\alpha\, dl}{4\pi^2\varepsilon_0 R^2}. \quad (19.13)$$

The first stage (finding the differential of the sought quantity) has been completed. Let us go on to the second (integration, summation). We must select the variable of integration. The position of a point charge on the semicircle is defined by angle α, so it is natural to select this angle as the integration variable. By definition angle α is defined as the ratio of the length of the arc l to the radius R of the circle: $\alpha = l/R$. Since $dl = R\, d\alpha$, we have

$$dE_x = \frac{Q \cos\alpha}{4\pi^2\varepsilon_0 R^2}\, d\alpha.$$

Integrating this equation with respect to angle α yields

$$E = \int_{-\pi/2}^{+\pi/2} \frac{Q}{4\pi^2\varepsilon_0 R^2} \cos\alpha \, d\alpha = \frac{Q}{2\pi^2\varepsilon_0 R^2}, \quad E \approx 1.4 \text{ V/m}. \tag{19.14}$$

We see that the correct result (19.14) differs considerably from the incorrect one (19.12). If we transform formula (19.14) by introducing the linear density of the electric charge carried by the semicircle, $\gamma = Q/\pi R$, we find that the strength of the electric field at the center of a uniformly charged arc in the form of a semicircle,

$$E = \frac{\gamma}{2\pi\varepsilon_0 R},$$

is given by the same formula as the strength of the electric field generated by an infinitely long and uniformly charged string (19.4).

The same method can be used to calculate the field generated by a uniformly charged ring (half-ring, etc.) at any point lying on its axis at a distance x from the ring's plane. Having calculated the field of a ring, we can formulate the problem of the motion of a charged particle along the ring's axis; for $x \ll R$ this motion constitutes harmonic oscillations. After this is done we can formulate the problem of calculating the electric field generated by a uniformly charged hemisphere, a part of a sphere, etc.

Example 19.5. *At the center of a hemisphere uniformly charged with electricity with a surface charge density* σ *there is placed a freely oriented point dipole with an electric moment* p. *Determine the potential energy of the dipole and the period of the dipole's small oscillations about an axis perpendicular to the symmetry axis of the hemisphere. The moment of inertia of the dipole about the rotation axis is* J.

Solution. The physical system consists of the uniformly charged hemisphere and the dipole. Neither can be thought of as a particle. A dipole is said to be a *point* dipole if its length is so small that in any nonhomogeneous field (the field generated by a hemisphere is nonhomogeneous) the torque acting on the dipole can be calculated by the

formula

$$\mathbf{M} = \mathbf{p} \times \mathbf{E}, \tag{19.15}$$

where **p** is the electric dipole moment. As is known, in a homogeneous field this formula holds true for any dipole.

To solve the problem, we must first calculate the field of the hemisphere (its electric field vector **E**) at the hemisphere's center. We apply the DI method. We partition the hemisphere into narrow rings and consider one ring (Figure 19.6). The charge of the ring is $dQ = 2\pi R^2\sigma \times \sin\theta \, d\theta$, where R is the radius of the sphere.

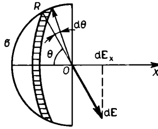

Figure 19.6

The projection of the elementary strength vector **dE** of the field generated by the ring on the X axis (the symmetry axis of the hemisphere) at point O is

$$dE_x = \frac{dQ}{4\pi\varepsilon_0 R^2}\cos\theta = \frac{\sigma \sin\theta\cos\theta \, d\theta}{2\varepsilon_0}. \tag{19.16}$$

Integrating this equation with respect to θ from $\theta_1 = 0$ (the farthest ring) to $\theta_2 = \pi/2$ (the closest ring), we find that

$$E = \int_0^{\pi/2} \frac{\sigma \sin\theta\cos\theta \, d\theta}{2\varepsilon_0} = \frac{\sigma}{4\varepsilon_0}. \tag{19.17}$$

Since the torque (19.15) acting on the dipole is known, we can use the equation of motion (14.7) to obtain the differential equation of the small oscillations of the dipole:

$$J\ddot{\varphi} = -pE\varphi, \quad \text{or} \quad \ddot{\varphi} + \omega_0^2\varphi = 0,$$

where $\omega_0^2 = pE/J$. The sought period is

$$T = 4\pi\sqrt{\varepsilon_0 J/p\sigma}.$$

To calculate symmetric fields (the electric field strength of an infinitely long cylindrical surface, of an infinite

cylinder, of a sphere, etc.) it has proved expedient to employ Gauss' law of flux.

Example 19.6. *A straight infinitely long cylinder of radius* $R_0 = 10$ cm *is uniformly charged with electricity with a surface charge density* $\sigma = +10^{-12}$ C/m². *The cylinder serves as a source of electrons, with the velocity vector of the emitted electrons perpendicular to its surface. What must the electron velocity be to ensure that the electrons can move away from the axis of the cylinder to a distance greater than* $r = 10^3$ m?

Solution. The physical system consists of two objects: the positively charged cylinder and an electron. The physical phenomenon consists of the electron moving in a decelerated manner in the electric field of the cylinder. We wish to find one of the parameters of motion, the electron velocity.

To describe the motion of the electron we must first calculate the electric field of the cylinder. The charge on the cylinder cannot be considered a point charge. We apply Gauss' law. For this we surround the cylinder with a cylindrical surface (coaxial with the cylinder) of arbitrary radius $r > R_0$ (Figure 19.7). In view of the symmetry of the problem, the electric vector **E** of the field of the cylinder is perpendicular at all points to the constructed cylindrical surface. Hence, the flux of **E** out of the cylindrical surface of length L is

$$\Phi_E = 2\pi rLE.$$

By Gauss' theorem,

$$2\pi rLE = 2\pi R_0 L\sigma/\varepsilon_0,$$

whence

$$E = \frac{R_0\sigma}{\varepsilon_0 r}. \tag{19.18}$$

Now, by applying the dynamical method we find that Newton's second law yields

$$m_e \frac{d^2r}{dt^2} = -e\frac{R_0\sigma}{\varepsilon_0 r},$$

where m_e is the electron mass, and e the electron charge. From the standpoint of physics the problem is solved.

It would be solved completely if we were to solve the above differential equation and obtain the law of motion of the electron $r = r(t)$. Knowing this law, we could

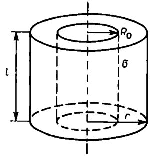

Figure 19.7

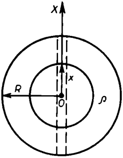

Figure 19.8

find the law of variation of the electron's velocity with time, $v = \dot{r}(t)$, and so on. But instead let us apply the law of energy conservation. By this law,

$$\frac{m_e v_0^2}{2} - e\varphi_0 = -e\varphi, \tag{19.19}$$

where φ_0 is the potential of the cylinder, and φ the potential of the field of the cylinder at a point r distant from the cylinder's axis. Employing the relationship $E = -d\varphi/dr$ that exists between the field strength E and potential φ and allowing for (19.18), we arrive at the following differential equation:

$$\frac{R_0\sigma}{\varepsilon_0 r} = -\frac{d\varphi}{dr}.$$

Integrating, we find that

$$\varphi = -\frac{R_0\sigma}{\varepsilon_0}\ln r + c, \tag{19.20}$$

with c an arbitrary constant. Hence,

$$\varphi_0 = -\frac{R_0\sigma}{\varepsilon_0}\ln R_0 + c. \tag{19.21}$$

The system of equations (19.19), (19.20), and (19.21) yields the following value for the sought initial velocity

of the electron:

$$v_0 = \sqrt{\frac{2eR_0\sigma \ln (r/R_0)}{\varepsilon_0 m_e}}, \quad v_0 \approx 3.7 \times 10^5 \text{ m/s}.$$

Concluding this section, we consider the following problem.

Example 19.7. *A solid ball made of an insulator ($\varepsilon \approx 1$) has been drilled along the diameter and air has been removed from the cavity. An electron is placed in the cavity. What is the magnitude of the positive charge that should be imparted to the ball if we want the ball to perform harmonic oscillations in the cavity with a given frequency ν_0 (the charge is assumed to be evenly distributed over the ball's volume)? Assume that the cross-sectional area A of the cavity is considerably smaller than πR^2, with R the radius of the ball.*

Solution. The problem is similar to Example 12.1 concerned with oscillations of an object in a shaft dug along Earth's diameter. To solve it we must calculate the electric field strength inside the ball. Let us apply Gauss' law. Suppose that the volume density of the charge, ρ, is equal to $3Q/4\pi R^3$. We take an arbitrary point x distant from the center of the ball and draw a sphere of radius x centered at the ball's center O and passing through that point (Figure 19.8). The flux of vector **E** out of the sphere is, in view of the symmetry of the field,

$$\Phi_E = E \times 4\pi x^2.$$

By Gauss' law,

$$E \times 4\pi x^2 = \frac{4\pi x^3 \rho}{3\varepsilon_0},$$

whence

$$E = \frac{\rho}{3\varepsilon_0} x.$$

Thus, the force acting on the electron is

$$F = -\frac{\rho e}{3\varepsilon_0} x.$$

From Newton's second law we get the differential equation of the electron's harmonic oscillations:

$$m_e\ddot{x} = -\frac{\rho e}{3\varepsilon_0}x.$$

Consequently, the angular frequency ω_0 is equal to $\sqrt{\rho e/3\varepsilon_0 m_e}$. Since $\omega_0 \equiv 2\pi\nu_0$, we can find the sought volume charge density,

$$\rho = 12\pi^2\varepsilon_0\nu_0^2 m_e/e,$$

and the charge on the ball,

$$Q = \frac{4}{3}\pi R^3\rho.$$

For $\nu_0 = 10^6$ Hz $= 1$ MHz and $R = 10^{-1}$ m we have $\rho \approx 6 \times 10^{-9}$ C/m^3 and $Q \approx 2.4 \times 10^{-11}$ C.

20. The Electrostatic Field in Insulators

When considering the electrostatic field in insulators (dielectrics), one uses Gauss' law of flux,

$$\oint_A \mathbf{D}\cdot d\mathbf{A} = \sum Q_i, \tag{20.1}$$

where

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \tag{20.2}$$

is the electric displacement vector, $\sum Q_i$ the sum of the free charges lying within the closed surface A, and $\mathbf{P}$ the polarization vector.

The strength $\mathbf{E}$ of an electric field in an insulator and the polarization $\mathbf{P}$ are interconnected through the following relation:

$$\mathbf{P} = \varepsilon_0(\varepsilon - 1)\mathbf{E}. \tag{20.3}$$

Thus,

$$\mathbf{D} = \varepsilon_0\varepsilon\mathbf{E}. \tag{20.4}$$

By the superposition principle, the electric field $\mathbf{E}$ in an insulator is the vector sum of the electric field generated by the free charges, $\mathbf{E}_0$, and by the bound charges, $\mathbf{E}'$:

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}'. \tag{20.5}$$

The surface density of the bound charges is

$$\sigma' = P_n = \varepsilon_0 (\varepsilon - 1) E_n, \qquad (20.6)$$

with P_n and E_n the normal components of the polarization and the field strength.

In calculating the field it is advisable to use one of the following two methods.

The first is based on the superposition principle (20.5); for the sake of brevity we will call it the *superposition method*. First the field generated by the free charges (sometimes called extraneous charges), $\mathbf{E}_0$, is calculated. Then the field $\mathbf{E}'$ generated by the bound charges is calculated. Finally, (20.5) is used to find the electric field vector in the insulator. This enables us to find the expression for the potential φ of the field in the insulator. Not all is as simple in this method as it might seem at first glance. Often the DI method must be used (see Section 6), and difficulties emerge in determining the density of the bound charges σ' (according to (20.6) this depends on E_n, which is also unknown) and the field $\mathbf{E}'$ generated by these charges, and other subtleties, which will be discussed later.

The second method uses Gauss' law (20.1) to find the electric displacement vector $\mathbf{D}$. Then (20.4) is employed to determine the electric field vector $\mathbf{E}$ in the insulator. If necessary, potential φ is calculated via (16.6). *Gauss' method* (for the sake of brevity we use this name to designate the second method) often leads to results faster and more simply than the superposition method, but sometimes Gauss' method proves inapplicable, while the superposition method can be applied even in such cases.

Note that in the majority of problems in this section the following conditions are assumed met: the insulators are homogeneous and isotropic and their boundaries coincide with equipotential surfaces.

Example 20.1. *A charge $Q = 10^{-9}$ C is imparted to one of the plates of a plane-parallel capacitor with a surface area $A = 0.2$ m² (the other plate is grounded). The distance between the plates is $d = 2$ mm. A glass plate and a porcelain plate are inserted between the plates of the capacitor*

(parallel to them), with the thickness of the first being $d_1 =$ 0.5 mm *and that of the second* $d_2 = 1.5$ mm, *so that there is no air gap between them. Determine the electric field*

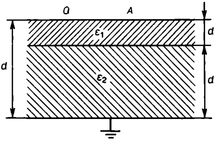

Figure 20.1

strength in each plate and the surface densities σ' *and* σ'' *of the bound charges carried by these plates (Figure 20.1).*

Solution. The physical system consists of the capacitor, whose plates carry free electric charges with a density $\sigma = Q/A$, and the two insulators, on which bound electric charges with densities σ' and σ'' appear. We wish to determine the electric field strengths E_1 and E_2 in the insulators and the densities σ' and σ'' of the bound charges. This constitutes a basic problem of field theory. We apply both methods.

The superposition method. In each insulator the field is generated by the free charges carried by the two parallel capacitor plates and, respectively, by the bound charges σ' and σ'' also carried by two planes. Note that the bound charges generate a field that is nonzero only inside "its own" insulator. Obviously, the electric fields generated by all types of charges are

$$E_0 = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}, \quad E_1' = \frac{\sigma'}{\varepsilon_0}, \quad E_2' = \frac{\sigma''}{\varepsilon_0}.$$

Since according to (20.6)

$$\sigma' = \varepsilon_0 (\varepsilon_1 - 1) E_1 \quad \text{and} \quad \sigma'' = \varepsilon_0 (\varepsilon_2 - 1) E_2,$$

and according to (20.5)

$$E_1 = E_0 - E_1' \quad \text{and} \quad E_2 = E_0 - E_2',$$

we solve the system of equations and obtain

$$E_1 = \frac{Q}{\varepsilon_0\varepsilon_1 A}, \quad E_2 = \frac{Q}{\varepsilon_0\varepsilon_2 A}, \quad \sigma' = \frac{(\varepsilon_1 - 1)Q}{\varepsilon_1 A}, \quad \sigma'' = \frac{(\varepsilon_2 - 1)Q}{\varepsilon_2 A}. \tag{20.7}$$

Gauss' method. Using Gauss' law, we can find the electric displacement vector in both insulators:

$$D\,\Delta A = \sigma\,\Delta A, \quad D = \sigma = Q/A.$$

Next, using (20.4), we can find the electric field strengths E_1 and E_2 in the insulators,

$$E_1 = \frac{Q}{\varepsilon_0\varepsilon_1 A}, \quad E_2 = \frac{Q}{\varepsilon_0\varepsilon_2 A},$$

and by employing (20.6) we can calculate the densities σ' and σ'' of the bound charges,

$$\sigma' = \frac{(\varepsilon_1 - 1)Q}{\varepsilon_1 A}, \quad \sigma'' = \frac{(\varepsilon_2 - 1)Q}{\varepsilon_2 A},$$

which coincides with the results (20.7) obtained by the superposition method.

Example 20.2. *Two infinitely long thin-walled coaxial cylinders of radii $R_1 = 5$ cm and $R_2 = 10$ cm have been uniformly charged electrically with surface densities $\sigma_1 = 10$ nC/m² and $\sigma_2 = -3$ nC/m². The space between the cylinders is filled with paraffin ($\varepsilon = 2$). Find the strength E of the field at points that lie at distances $r_1 = 2$ cm, $r_2 = 6$ cm, and $r_3 = 15$ cm from the axes of the cylinders.*

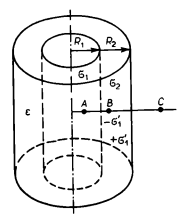

Figure 20.0

Solution. (1) *The superposition method.* The total field is generated by four charges: the free charges with densities σ_1 and σ_2 and the bound charges with densities $-\sigma_1'$ and $+\sigma_1'$ (Figure 20.2). The bound charges generate a field

that is nonzero only inside the insulator (paraffin). Clearly, the field at point A (positioned at a distance $r_1 = 2$ cm from the axes) is zero (this can be proved by applying Gauss' law for a vacuum).

Let us now consider point B (positioned at a distance $r_2 = 6$ cm from the axes). At this point the field is generated by the charges with densities σ_1 and $-\sigma_1'$ (the field generated by the charges $+\sigma_1'$ and σ_2 at this point is zero). By Gauss' law (for a vacuum), the field generated by σ_1 is

$$E_1 = \frac{R_1\sigma_1}{\varepsilon_0}\frac{1}{r_2}. \tag{20.8}$$

In the same manner we can find the field generated by σ_1':

$$E_1' = \frac{R_1\sigma_1'}{\varepsilon_0}\frac{1}{r_2}. \tag{20.9}$$

According to (20.6),

$$\sigma_1' = \varepsilon_0(\varepsilon - 1)E(R_1). \tag{20.10}$$

It is important to note that $E(R_1)$ in (20.10) is the strength of the total electric field in the insulator at a point that is R_1 distant from the axes. This quantity is unknown, but let us relate it with $E(r_2)$, the total electric field in the insulator at point B. Since both E_1 and E_1' are inversely proportional to the distance r_2 from point B to the axes, the total field strengths $E(R_1)$ and $E(r_2)$ must obey the same condition:

$$\frac{E(R_1)}{E(r_2)} = \frac{r_2}{R_1}. \tag{20.11}$$

Hence,

$$E_1' = (\varepsilon - 1)E(r_2).$$

Since according to (20.5)

$$E(r_2) = E_1 - E_1',$$

we get

$$E(r_2) = \frac{R_1\sigma_1}{\varepsilon_0\varepsilon}\frac{1}{r_2} \tag{20.12}$$

This yields $E(r_2) \approx 4.7 \times 10^2$ V/m.

At point C lying at a distance $r_3 = 15$ cm from the axes, the field is generated only by the free charges with

densities σ_1 and σ_2:

$$E(r_3) = \frac{R_1\sigma_1}{\varepsilon_0}\frac{1}{r_3} - \frac{R_2\sigma_2}{\varepsilon_0}\frac{1}{r_3}.$$

This yields $E(r_3) \approx 1.5 \times 10^2$ V/m.

(2) *Gauss' method*. Let us first calculate the field at point B. By Gauss' law of flux,

$$D \times 2\pi r_2 l = 2\pi R_1 \sigma_1 l,$$

which yields $D = R_1\sigma_1/r_2$. Now, using (20.4), we can find the sought field:

$$E(r_2) = \frac{R_1\sigma_1}{\varepsilon_0\varepsilon}\frac{1}{r_2},$$

which coincides with the expression (20.12) obtained by the superposition method. Hence, $E(r_2) \approx 4.7 \times 10^2$ V/m.

If the field (i.e. the electric vector $\mathbf{E}$) is known, other quantities, such as the potential and the energy, can be found.

Example 20.3. *Two concentric metal spheres of radii* $R_1 = 4$ cm *and* $R_2 = 10$ cm *carry charges* $Q_1 = -2$ nC *and* $Q_2 = 3$ nC, *respectively. The space between the spheres is filled with ebonite* ($\varepsilon = 3$). *Determine the potential* φ *of the electric field at distances* $r_1 = 2$ cm, $r_2 = 6$ cm, *and* $r_3 = 20$ cm *from the common center of the spheres.*

Solution. (1) *The superposition method.* The total field is generated by the free charges Q_1 and Q_2 and the bound charges Q_1' and Q_1'' (Figure 20.3). To find Q_1' and Q_1'' we must know the field strength $E(r)$ in the insulator. This quantity can be found by applying Gauss' law:

$$E(r) = \frac{Q_1}{4\pi\varepsilon_0\varepsilon r^2}. \qquad (20.13)$$

Suppose that σ_1' and σ_1'' are the surface densities of the bound charges Q_1' and Q_1'', respectively. Then from (20.6) and (20.13) it follows that

$$\sigma_1' = \varepsilon_0(\varepsilon - 1)E_{n1} = \frac{(\varepsilon-1)Q_1}{4\pi\varepsilon R_1^2},$$

$$\sigma_1'' = \varepsilon_0(\varepsilon - 1)E_{n2} = \frac{(\varepsilon-1)Q_1}{4\pi\varepsilon R_2^2}.$$

Thus,

$$Q_1' = \sigma_1' 4\pi R_1^2 = \frac{(\varepsilon-1)Q_1}{\varepsilon}, \quad Q_1'' = \sigma_1'' 4\pi R_2^2 = \frac{(\varepsilon-1)Q_1}{\varepsilon}. \tag{20.14}$$

It is known that for a uniformly charged sphere of radius R the potential of the field (in a vacuum) inside the

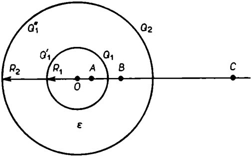

Figure 20.3

sphere and on its surface is

$$\varphi = \frac{Q}{4\pi\varepsilon_0 R}, \tag{20.15}$$

while at points outside the sphere the potential is

$$\varphi = \frac{Q}{4\pi\varepsilon_0 r}, \tag{20.16}$$

where Q is the charge on the sphere, and r the distance from the center of the sphere, O, to the point in question. Since the charges (Q_1, Q_2, Q_1', and Q_1'') generating the field are distributed over spherical surfaces, by allowing for the superposition principle and Eqs. (20.14), (20.15), and (20.16) we can determine potential φ_{01} at the point A positioned at a distance r_1 from the center:

$$\varphi_{01} = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$$
$$= \frac{-Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_1'}{4\pi\varepsilon_0 R_1} + \frac{-Q_1''}{4\pi\varepsilon_0 R_2} + \frac{Q_2}{4\pi\varepsilon_0 R_2}$$

$$= -\frac{Q_1}{4\pi\varepsilon_0 R_1}+\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon R_1}-\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon R_2}+\frac{Q_2}{4\pi\varepsilon_0 R_2}$$

$$= -\frac{Q_1}{4\pi\varepsilon_0\varepsilon R_1}-\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon R_2}+\frac{Q_2}{4\pi\varepsilon_0 R_2}; \qquad (20.17)$$

the potential φ_{02} at the point B positioned at a distance r_2 from the center:

$$\varphi_{02} = -\frac{Q_1}{4\pi\varepsilon_0 r_2}+\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon r_2}-\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon R_2}+\frac{Q_2}{4\pi\varepsilon_0 R_2}$$

$$= -\frac{Q_1}{4\pi\varepsilon_0\varepsilon r_2}-\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon R_2}+\frac{Q_2}{4\pi\varepsilon_0 R_2}; \qquad (20.18)$$

and the potential φ_{03} at the point C positioned at a distance r_3 from the center:

$$\varphi_{03} = -\frac{Q_1}{4\pi\varepsilon_0 r_3}+\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon r_3}-\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon r_3}+\frac{Q_2}{4\pi\varepsilon_0 r_3}$$

$$= -\frac{Q_1}{4\pi\varepsilon_0 r_3}+\frac{Q_2}{4\pi\varepsilon_0 r_3}. \qquad (20.19)$$

(2) *Gauss' method*. Using Gauss' law, we find the field strengths E_{01}, E_{02}, and E_{03} at points A, B, and C, respectively:

$$E_{01}=0, \quad E_{02}=\frac{Q_1}{4\pi\varepsilon_0\varepsilon r_2^2}, \quad E_{03}=\frac{Q_2-Q_1}{4\pi\varepsilon_0 r_3^2}. \qquad (20.20)$$

Since in our case the potential φ is continuous, employing the relationship (16.6) between field strength and potential we can find the value of the potential at any point if we know the potential at least at one point. This potential is φ_{03} given by (20.19) because it is generated by the free charges Q_1 and Q_2 or the potential at the surface of the second sphere,

$$\varphi(R_2)=\frac{Q_2-Q_1}{4\pi\varepsilon_0 R_2}. \qquad (20.21)$$

Integrating (20.13) with respect to r from r_2 to R_2, we obtain

$$+\varphi(R_2)-\varphi(r_2)=\frac{Q_1}{4\pi\varepsilon_0\varepsilon r_2}-\frac{Q_1}{4\pi\varepsilon_0\varepsilon R_2}.$$

This yields $\varphi(r_2)\equiv\varphi_{02}$:

$$\varphi_{02}=-\frac{Q_1}{4\pi\varepsilon_0\varepsilon r_2}-\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon R_2}+\frac{Q_2}{4\pi\varepsilon_0 R_2}.$$

which coincides with the expression (20.18) obtained by the superposition method.

Integrating (20.13) from R_1 to r_2, we find that

$$\varphi(r_2)-\varphi(R_1)=\frac{Q_1}{4\pi\varepsilon_0\varepsilon R_1}-\frac{Q_1}{4\pi\varepsilon_0\varepsilon r_2}.$$

This yields $\varphi(R_1)\equiv\varphi_{01}$:

$$\varphi_{01}=-\frac{Q_1}{4\pi\varepsilon_0\varepsilon R_1}-\frac{(\varepsilon-1)Q_1}{4\pi\varepsilon_0\varepsilon R_2}+\frac{Q_2}{4\pi\varepsilon_0 R_2},$$

which coincides with the expression (20.17) obtained by the superposition method.

This problem can be made more complicated if we add one or several charged concentric spheres and place different insulators between the spheres and inside the first sphere. Clearly, all these problems can be solved either by the superposition method or Gauss' method.

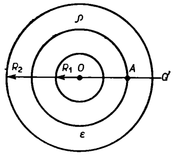

Figure 20.4

Let us now consider a problem that, if solved by the superposition method, requires special care, with the precise analysis and calculation of the fields generated by different charges.

Example 20.4. *A thick-walled hollow ball of glass* ($\varepsilon=7$) *is uniformly charged over its volume with a density* $\rho=1.5\ \mu C/m^3$. *The inner radius of the ball is* $R_1=2$ cm *and the outer* $R_2=6$ cm. *Find the distribution of the potential in the glass and calculate the potential at the outer and inner surfaces of the ball and at its center.*

Solution. (1) *The superposition method.* Let us first find the distribution of the potential in the glass of the ball, that is, the potential at an arbitrary point A that is r distant from the ball's center with $R_1<r<R_2$ (Figure 20.4). What charges generate the field in the insulator?

First, this is the bound space charge of the insulator

$$Q = \frac{4}{3}\pi (R_2^3 - R_1^3)\rho \qquad (20.22)$$

and, second, the bound charge Q' on the outer surface of the ball, which according to (20.14) is

$$Q' = \frac{(\varepsilon - 1) Q}{\varepsilon}, \qquad (20.23)$$

Let us find at point A the potential of the field generated by the bound space charge Q. Since this is not a point charge, we must employ the DI method. But if we apply this method formally, we may arrive at a wrong result. In view of the symmetric distribution of charge Q, we partition the hollow ball into thin concentric spherical layers whose thickness dx is so small that the elementary charge

$$dQ = 4\pi x^2 \, dx\rho \qquad (20.24)$$

(where x is the radius of the layer) of each layer can be thought of as distributed over a sphere. Then, according to the superposition principle, the total potential at point A is equal to the sum of the potentials of the fields generated by the elementary charges on these spheres. But the elementary potential generated at point A by a sphere must be calculated by different formulas, (20.15) or (20.16), depending on whether point A is an interior or exterior point in relation to the particular sphere.

To allow for this complication, we draw through point A a sphere of radius r centered at point O (see Figure 20.4). This sphere divides the hollow ball into two sublayers: a sublayer with radii R_1 and r and a sublayer with radii r and R_2. In relation to the first sublayer, point A is always an exterior point with respect to the elementary spheres of the sublayer, while in relation to the second, point A is always an interior point. Consider an elementary spherical layer of thickness dx in the first sublayer (Figure 20.5). Its elementary potential at point A is, according to (20.16),

$$d\varphi = \frac{dQ}{4\pi\varepsilon_0 \varepsilon r}, \qquad (20.25)$$

where we have allowed for the fact that the charge dQ is located in an insulator.

The validity of (20.25) in the superposition method is justified in the following manner. The potential at point A is generated by the free charge dQ and the bound charge $-dQ' = -[(\varepsilon-1)/\varepsilon]\, dQ$ distributed over the sphere of

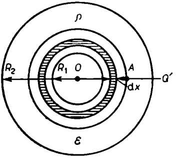

Figure 20.5

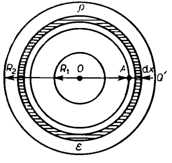

Figure 20.6

radius x. The total potential generated by these charges at point A is

$$d\varphi = \frac{dQ}{4\pi\varepsilon_0 r} - \frac{dQ'}{4\pi\varepsilon_0 r} = \frac{dQ}{4\pi\varepsilon_0 r} - \frac{(\varepsilon-1)\, dQ}{4\pi\varepsilon_0 \varepsilon r} = \frac{dQ}{4\pi\varepsilon_0 \varepsilon r},$$

which coincides with (20.25).

Integrating (20.25) with respect to x from R_1 to r, we find the potential φ_1 generated at point A by the charge of the first sublayer:

$$\varphi_1 = \int_{R_1}^{r} \frac{4\pi\rho x^2\, dx}{4\pi\varepsilon_0 \varepsilon r} = \frac{\rho}{3\varepsilon_0\varepsilon}\left(r^2 - \frac{R_1^3}{r}\right). \qquad (20.26)$$

We now consider the second sublayer and take an elementary spherical layer of thickness dx in this sublayer (Figure 20.6). The elementary potential generated at point A by the charge on this spherical layer is, according to (20.15),

$$d\varphi = \frac{dQ}{4\pi\varepsilon_0 \varepsilon x}. \qquad (20.27)$$

Allowing for (20.24) and integrating (20.27) with respect to x from r to R_2, we find the potential φ_2 generated at point A by the charge of the second sublayer:

$$\varphi_2 = \int_r^{R_2} \frac{4\pi\rho x^2\,dx}{4\pi\varepsilon_0\varepsilon x} = \frac{\rho R_2^2}{2\varepsilon_0\varepsilon} - \frac{\rho r^2}{2\varepsilon_0\varepsilon}. \qquad (20.28)$$

Finally, the potential φ_3 generated at point A by the bound charge Q' (see (20.23)) is, according to (20.15),

$$\varphi_3 = \frac{(\varepsilon-1)(R_2^3-R_1^3)\rho}{3\varepsilon_0\varepsilon R_2}. \qquad (20.29)$$

Thus, the total potential generated at point A by all charges is

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 = -\frac{\rho}{3\varepsilon_0\varepsilon}\left(\frac{r^2}{2} + \frac{R_1^3}{r}\right) + C, \qquad (20.30)$$

with

$$C = \frac{\rho R_2^2}{2\varepsilon_0\varepsilon} + \frac{(\varepsilon-1)(R_2^3-R_1^3)\rho}{3\varepsilon_0\varepsilon R_2} \qquad (20.31)$$

a constant. Let us transform C to a form needed later:

$$C = \frac{(R_2^3-R_1^3)\rho}{3\varepsilon_0 R_2} + \frac{\rho(R_2^3+2R_1^3)}{6\varepsilon_0\varepsilon R_2}. \qquad (20.32)$$

Knowing the potential distribution (20.30), we can find the potential at the outer surface of the ball,

$$\varphi(R_2) = -\frac{\rho}{3\varepsilon_0\varepsilon}\left(\frac{R_2^2}{2} + \frac{R_1^3}{R_2}\right) + C,$$

and at the inner surface,

$$\varphi(R_1) = -\frac{\rho R_1^2}{2\varepsilon_0\varepsilon} + C.$$

The potential at the center of the ball, φ_0, is equal to the potential at the inner surface, $\varphi(R_1)$.

(2) *Gauss' method.* Using Gauss' law, we find the magnitude of the electric displacement vector at point A (see Figure 20.6):

$$D \times 4\pi r^2 = \frac{4}{3}\pi(r^3 - R_1^3)\rho,$$

$$D = \frac{\rho}{3}\left(r - \frac{R_1^3}{r^2}\right). \qquad (20.33)$$

Next we find the magnitude of the electric field strength at the same point:

$$E=\frac{\rho}{3\varepsilon_0\varepsilon}\left(r-\frac{R_1^3}{r^2}\right). \tag{20.34}$$

Upon integrating the relationship $E=-d\varphi/dr$ we get the potential distribution in the ball:

$$\varphi=-\int\frac{\rho}{3\varepsilon_0\varepsilon}\left(r-\frac{R_1^3}{r^2}\right)dr=-\frac{\rho}{3\varepsilon_0\varepsilon}\left(\frac{r^2}{2}+\frac{R_1^3}{r}\right)+C_1. \tag{20.35}$$

The integration constant C_1 will be found from the requirement that the potential be continuous and the fact that the potential at the outer surface of the ball is determined solely by the value of the free charge Q (20.22):

$$\varphi(R_2)=\frac{Q}{4\pi\varepsilon_0R_2}=\frac{(R_2^3-R_1^3)\rho}{3\varepsilon_0R_2}.$$

Substituting this into (20.35),

$$\frac{(R_2^3-R_1^3)\rho}{3\varepsilon_0R_2}=-\frac{\rho}{3\varepsilon_0\varepsilon}\left(\frac{R_2^2}{2}+\frac{R_1^3}{R_2}\right)+C_1,$$

we arrive at the following expression for the integration constant:

$$C_1=\frac{(R_2^3-R_1^3)\rho}{3\varepsilon_0R_2}+\frac{(R_2^3+2R_1^3)\rho}{6\varepsilon_0\varepsilon R_2}. \tag{20.36}$$

Thus, the final result is

$$\varphi=-\frac{\rho}{3\varepsilon_0\varepsilon}\left(\frac{r^2}{2}+\frac{R_1^3}{r}\right)+\frac{(R_2^3-R_1^3)\rho}{3\varepsilon_0R_2}+\frac{(R_2^3+2R_1^3)\rho}{6\varepsilon_0\varepsilon R_2}, \tag{20.37}$$

which coincides with the expression (20.30) obtained by the superposition method if we allow for the value of C (20.32).

The problem can be made more complicated if, say, we put in the cavity a metal ball or a dielectric ball (with a different dielectric constant ε_1) uncharged or charged over its volume with another charge density ρ_1, etc. For example, from (20.37) we can obtain an expression for the potential distribution inside a uniformly charged

solid ball of radius R,

$$\varphi = -\frac{\rho r^2}{6\varepsilon_0\varepsilon} + \frac{\rho R^2}{3\varepsilon_0}\left(1 + \frac{1}{2\varepsilon}\right),$$

and at the center of the ball,

$$\varphi(0) = \frac{\rho R^2}{3\varepsilon_0}\left(1 + \frac{1}{2\varepsilon}\right).$$

All these and similar problems can be solved by applying either the superposition method or Gauss' method.

The reader must have noticed that in all the problems solved in this section Gauss' method proved to be simpler and led to a result faster than the superposition method. However, let us not hurry with conclusions but rather consider a few more problems.

Example 20.5. *A sufficiently long, round cylinder made from a homogeneous and isotropic insulator with a known dielectric constant* ε *is placed in a homogeneous electric field* $\mathbf{E}_0$ *in such a manner that the cylinder's axis coincides*

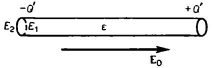

Figure 20.7

with the direction of $\mathbf{E}_0$ *(Figure 20.7). Determine the electric field strength near the cylinder (inside and outside).*

Solution. Clearly, Gauss' method is useless here. Applying Gauss' law, we arrive at the trivial identity $D_1 = D_2$ expressing the continuity of the normal components of the electric displacement vector. Let us apply the superposition method. By E_1 we denote the electric field strength inside the cylinder and by E_2 the electric field strength outside. Owing to the polarization of the insulator, bound charges $-Q'$ and $-Q'$ gather on the bases of the cylinder with a density σ'. The resulting electric fields E_1 and E_2 are the vector sums of E_0 and the electric fields generated by the bound charges $-Q'$ and $+Q'$.

Let us now discuss the meaning of the words "sufficiently long cylinder". The cylinder considered here is so long that the field generated, say, by charge $+Q'$ is weak in the vicinity of charge $-Q'$ and can be neglected in comparison to the field generated by $-Q'$ in that vicinity. The same is true of the field generated by $-Q'$ in the vicinity of charge $+Q'$. Thus,

$$E_1 = E_0 - E', \quad E_2 = E_0 + E',$$

where E' is the electric field strength generated by $-Q'$ (or $+Q'$). Let us find E'.

E' is the field of a uniformly charged disk. Applying the DI method, we find (see Figure 20.7, Example (18.5), and formula (18.20)) the projection of the elementary electric field vector on the disk's axis generated by a thin ring (the X axis is directed along the axis of the disk):

$$\mathrm{d}E_x = \frac{x\,\mathrm{d}Q}{4\pi\varepsilon_0 (r^2+x^2)^{3/2}} = \frac{3\pi r\sigma' x\,\mathrm{d}r}{4\pi\varepsilon_0 (r^2+x^2)^{3/2}}.$$

Integration with respect to r from zero to R (the radius of the disk) yields the electric field strength generated by the disk (or the field of the bound charge $-Q'$):

$$E' = E_x = \int\limits_0^R \frac{\sigma' xr\,\mathrm{d}r}{2\varepsilon_0 (r^2+x^2)^{3/2}} = \frac{\sigma'}{2\varepsilon_0}\left[1 - \frac{x}{\sqrt{x^2+R^2}}\right]. \tag{20.38}$$

From this it follows, for one, that E' is roughly zero when x is very large. This completes the justification for using the term "sufficiently long cylinder".

Near the base of the cylinder $x \approx 0$ and

$$E' = \sigma'/(2\varepsilon_0). \tag{20.39}$$

Allowing for (20.6), we obtain

$$E_1 = E_0 - \frac{\varepsilon_0(\varepsilon-1)E_1}{2\varepsilon_0}$$

and, hence,

$$E_1 = \frac{2}{1+\varepsilon} E_0. \tag{20.40}$$

Note that the E_n in (20.6) is the electric field strength inside the insulator. In our case $E_n = E_1$. Then (20.6) yields

$$\sigma' = 2\varepsilon_0 \frac{\varepsilon-1}{\varepsilon+1} E_0. \qquad (20.41)$$

From (20.39) we obtain

$$E' = \frac{\varepsilon-1}{\varepsilon+1} E_0$$

Hence,

$$E_2 = \frac{2\varepsilon}{1+\varepsilon} E_0. \qquad (20.42)$$

Example 20.6. *An infinite homogeneous, isotropic insulator in which a specified homogeneous electric field E_0 has been created contains a spherical cavity of radius R (Figure 20.8). A point dipole with an electric moment* **p** *is placed at the center of the cavity. Determine the period of the dipole's small oscillations if the moment of inertia of the dipole about the rotation axis is J.*

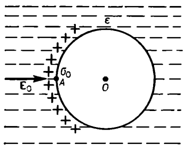

Figure 20.8

Solution. The problem is similar to Example 19.5. We can easily find the period of the dipole's small oscillations if we know the field in the cavity. Clearly, Gauss' method is useless here. Let us apply the superposition method. Owing to the polarization of the insulator, bound charges $+Q'$ and $-Q'$ gather on the two hemispheres, and the densities σ' of these charges are not constants. To calculate the field generated by a charged hemisphere with a variable surface charge density σ' let us employ the DI method. It can easily be demonstrated that on an elementary ring the surface charge density is

$$\sigma' = \sigma_0 \cos\theta, \qquad (20.43)$$

where

$$\sigma_0 = \varepsilon_0 (\varepsilon - 1) E_0 \qquad (20.44)$$

is the maximum surface charge density (at point A in Figure 20.8). Then the projection of the elementary electric field vector $d\mathbf{E}$ generated by the ring at point O on the X axis (the symmetry axis of the hemisphere) is, according to (19.16),

$$dE_x = \frac{\sigma_0 \sin\theta \cos^2\theta \, d\theta}{2\varepsilon_0}.$$

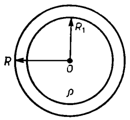

Figure 20.9

Integration yields the electric field generated by one charged hemisphere:

$$E_1 = \int_0^{\pi/2} \frac{\sigma_0 \sin\theta \cos^2\theta \, d\theta}{2\varepsilon_0} = \frac{(\varepsilon - 1) E_0}{6}.$$

The field generated by the two hemispheres is

$$E_2 = 2E_1 = \frac{(\varepsilon - 1) E_0}{3}.$$

Thus, the sought field at the center of the spherical cavity is

$$E = E_0 + E_2 = \frac{2 + \varepsilon}{3} E_0. \qquad (20.45)$$

Since the field is now known, the subsequent solution of the problem on the oscillations of the dipole is obvious (see Example 19.5).

To conclude this section we examine another problem involving estimation.

Example 20.7. *An ebonite ball of radius R is uniformly charged with electricity with a volume density ρ. What is the radius R_1 of the sphere that divides the ball into two parts whose energies are equal.*

Solution. Let us draw a sphere of radius R_1 (Figure 20.9). We must determine the energy W_1 of a ball of radius R_1 and the energy W_2 of a spherical layer with radii R_1 and R. For this we must know the field in the ball. This can easily be found by employing Gauss' method. By

Gauss' law of flux,

$$D \times 4\pi r^2 = \frac{4}{3}\pi r^3 \rho.$$

Thus, the field inside the ball is

$$E = \frac{\rho r}{3\varepsilon_0 \varepsilon}.$$

Applying the DI method, we find the energy $\mathrm{d}W$ of the field existing inside a thin spherical layer of thickness $\mathrm{d}r$:

$$\mathrm{d}W = w\,\mathrm{d}V = \frac{\varepsilon_0 \varepsilon E^2}{2} 4\pi r^2\,\mathrm{d}r = \frac{2\pi\rho^2}{9\varepsilon_0 \varepsilon} r^4\,\mathrm{d}r,$$

where w is the energy density of the electric field. Integrating, we obtain

$$W_1 = \frac{2\pi\rho^2 R_1^5}{45\varepsilon_0 \varepsilon}, \quad W_2 = \frac{2\pi\rho^2 (R^5 - R_1^5)}{45\varepsilon_0 \varepsilon}.$$

Since $W_1 = W_2$, we find that

$$R_1 = \frac{R}{\sqrt[5]{2}} \approx 0.87R.$$

The numerical answer is somewhat unexpected: the outer spherical layer whose thickness is only (approximately) one-tenth of the radius contains half of the energy of the entire ball.

21. Conductors in an Electrostatic Field

The surface of a conductor constitutes an equipotential surface. The *method of images* is based on this property. This method makes it possible to calculate various electrostatic fields, determine the capacitance of a system of conductors, etc.

The method of images relies on the following statement: if in an arbitrary electrostatic field we replace an equipotential surface with a metal surface of the same shape and create the same potential on the metal surface, the electrostatic field does not change.

Let us consider the electric field that exists in the space between a point charge $+Q$ and an infinite metal plane whose potential is zero. In view of the above principle this field is equivalent to an electric field generated by the

given point charge $+Q$ and a point charge $-Q$ that is the mirror image of the given charge $+Q$ in the metal plane (Figure 21.1).

Example 29.1. *A point charge $Q = -2 \times 10^{-8}$ C is placed at a distance $l = 1$ m from an infinite metal plane that is grounded (Figure 21.1). Determine the interaction between the charge and the plane.*

Solution. The metal plane is in the electrostatic field of the point charge. Owing to electrostatic induction, on the side of the plane closest to the charge there appear induced

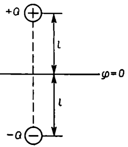

Figure 21.1

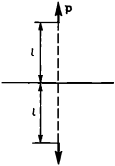

Figure 21.2

electric charges of the opposite sign. Hence, a force of interaction develops between the given point charge and the charges induced on the metal surface. By the hypothesis, the potential of the metal plane is zero (Earth's potential is assumed to be zero). Consequently, according to the method of images, the electric field existing in the space between the point charge and the metal surface is equivalent to the field generated by the given point charge and its mirror image in the metal plane. According to Coulomb's law, the sought interaction is given by the force

$$F = \frac{Q^2}{4\pi\varepsilon_0 (2l)^2}, \quad F \approx 9 \times 10^{-7}\ \text{N}.$$

Example 21.2. *A point dipole with an electric moment* **p** *is placed at a distance l from an infinite conducting plane.*

Determine the magnitude of the force acting on the dipole if vector **p** *is perpendicular to the plane.*

Solution. According to the method of images, the field generated by the given dipole and the charges induced on the plane is equivalent to the field generated by two dipoles, namely, the given dipole and its mirror image in the plane (Figure 21.2). The dipoles are separated by a distance of $2l$. Thus, the sought force F is that with which the image-dipole acts on the given dipole. It can easily be proved that the field strength E at a point on the dipole's axis at a distance $r \gg l_1$ (the dipole's length) is given by the following formula

$$E = \frac{2p}{4\pi\varepsilon_0 r^3}.$$

Hence, the force acting on the given dipole is

$$F = F_1 - F_2,$$

where

$$F_1 = QE_1 = \frac{2pQ}{4\pi\varepsilon_0 r^3}$$

is the force acting on the negative charge of the given dipole, and

$$F_2 = QE_2 = \frac{2pQ}{4\pi\varepsilon_0 (r + l_1)^3}$$

the force acting on the positive charge of the given dipole. Allowing for the fact that $r = 2l$ and $p = Ql_1$, $2l \gg l_1$ and performing simple manipulations, we obtain

$$F = \frac{3p^2}{32\pi\varepsilon_0 l^4}.$$

Example 21.3. *A thin, infinitely long string is uniformly charged with electricity with a linear density* τ *and is placed parallel to an infinite conducting plane at a distance* l *from the latter (Figure 21.3). Find (a) the magnitude of the force acting on a section of the string of unit length, and (b) the distribution of the surface charge density* $\sigma(x)$ *in the plane, where* x *is the distance from the plane that is perpendicular to the conducting plane and passes through the string.*

Solution. Clearly, to find the force acting on a segment of the string of unit length we must calculate the field generated by the mirror image of the string (see Figure 21.3). The field generated by the string can easily be

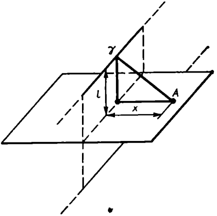

Figure 21.3

found, say by employing Gauss' law of flux (see (19.4)):

$$E = \frac{\tau}{2\pi\varepsilon_0 r}.$$

In our case $r = 2l$. Thus, the force acting on the segment of the string of unit length is

$$F = E\tau = \frac{\tau^2}{4\pi\varepsilon_0 l}.$$

To carry out the second task, we determine the field generated by the string and the string's image at point A (see Figure 21.3):

$$E_1 = \frac{2\tau l}{2\pi\varepsilon_0 (x^2+l^2)} = \frac{\tau l}{\pi\varepsilon_0 (x^2+l^2)}.$$

The induced charges generate a field E_2 near the conducting plane (outside it), which we find by employing Gauss' law:

$$E_2 = \sigma/\varepsilon_0. \tag{21.1}$$

These induced charges distribute themselves in such a manner that their field inside the plane neutralizes the effect of the external field E_1 (i.e. the field inside a conductor placed in an electrostatic field is nil):

$$\mathbf{E}_1 + \mathbf{E}_2 = 0. \qquad (21.2)$$

Consequently,

$$\sigma = \frac{\tau l}{\pi (x^2 + l^2)}.$$

Thus, in the method of images we are most often confronted with the problem of calculating the field characteristics of specified charges and their mirror images, that is, solving the basic problem of field theory.

Example 21.4. *A very long straight string has been uniformly charged with electricity with a linear density* τ *and is placed at right angles to an infinite conducting plane in such a way that its lower end is* l *distant from the plane (Figure 21.4). Point O is the trace of the string on the plane. Determine the surface density of the charge induced on the plane (a) at point O, and (b) at a point A that is* x *distant from point O (in the plane).*

Solution. Using the method of images, we first calculate the field generated by the string and its image at point O. To determine the field generated by the string alone at point O we employ the DI method. The point charge $dQ = \tau\, dr$ carried by an element of the string of length dr generates at an arbitrary point on the string's axis r distant from the element (see Figure 21.4) an electric field

$$dE = \frac{dQ}{4\pi\varepsilon_0 r^2} = \frac{\tau\, dr}{4\pi\varepsilon_0 r^2}.$$

Integration yields

$$E = \int_r^\infty \frac{\tau\, dr}{4\pi\varepsilon_0 r^2} = \frac{\tau}{4\pi\varepsilon_0 r}.$$

In our case $r = l$. Thus, the field strength E_1 generated at point O by the string and its image is

$$E_1 = 2E = \frac{\tau}{2\pi\varepsilon_0 l}.$$

Allowing for (21.1) and (21.2), we find the density σ_0 of the charges at point O in the conducting plane:

$$\sigma_0 = \frac{\tau}{2\pi l}.$$

Let us now find the density σ of the charges induced at point A in the plane (Figure 21.5). To do this we must again calculate the field of the string and its image, but this time the field generated at point A. Applying the

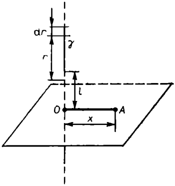

Figure 21.4

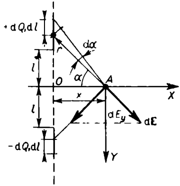

Figure 21.5

DI method, we find the magnitude of the vector dE representing the elementary field generated at point A by the point charge $dQ = \tau\, dl$ of the element dl (r distant from A) of the string alone:

$$dE = \frac{dQ}{4\pi\varepsilon_0 r^2} = \frac{\tau\, dl}{4\pi\varepsilon_0 r^2}.$$

Since $dl = r\, d\alpha/\cos\alpha$ and $r = x/\cos\alpha$, we have

$$dE = \frac{\tau\, d\alpha}{4\pi\varepsilon_0 x}.$$

Figure 21.5 shows that the resultant electric field generated by the string and its image at point A is directed along the Y axis and, hence, $E_x = 0$. Therefore, we will only find the projection dE_y of dE:

$$dE_y = dE \sin\alpha = \frac{\tau \sin\alpha\, d\alpha}{4\pi\varepsilon_0 x}.$$

Integration yields the projection of the electric field vector of the string on the Y axis:

$$E_y = \int_{\alpha_1}^{\pi/2} \frac{\tau \sin \alpha \, d\alpha}{4\pi\varepsilon_0 x} = \frac{\tau \cos \alpha_1}{4\pi\varepsilon_0 x}.$$

Thus, the electric field of the string and its image generated at point A:

$$E_1 = 2E_y = \frac{\tau \cos \alpha_1}{2\pi\varepsilon_0 x} = \frac{\tau}{2\pi\varepsilon_0 (l^2 + x^2)^{1/2}}.$$

Allowing for (21.1) and (21.2), we can determine the surface density of the charges induced on the plane:

$$\sigma = \frac{\tau}{2\pi (l^2 + x^2)^{1/2}}.$$

Clearly, for $x = 0$ we have $\sigma = \sigma_0$.

Example 21.5. *A thin ring of radius R has been uniformly charged with an amount of electricity Q and placed in relation to a conducting sphere in such a way that the center of the sphere, O, lies on the ring's axis at a distance of l from the plane of the ring (Figure 21.6). Determine the potential of the sphere.*

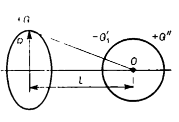

Figure 21.6

Solution. The conducting sphere is situated in the field of the ring. We wish to calculate the potential of the conductor. This constitutes a basic problem of field theory. Since the field is not symmetric, it is doubtful that Gauss' law of flux will lead to meaningful results. Let us employ the superposition method.

The field of the ring induces charges of magnitude $-Q'$ and $+Q''$ on the conducting sphere. The resultant field is generated by three charges: Q, $-Q'$, and $+Q''$. Hence, according to the superposition principle, the potential of the conducting sphere is $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, where φ_1, φ_2, and φ_3 are the potentials of the fields generated by the

charges Q, $-Q'$, and $+Q''$, respectively. But at what point of the sphere? The answer is: at any point, since the potential of a conductor placed in an electrostatic field is the same for all points of the conductor.

In our case, the entire volume bound by the conducting sphere is equipotential. Thus, we need only calculate the potential at the most convenient point, the center of the sphere. Indeed, notwithstanding the fact that we know neither the values of the induced charges $-Q'$ and $+Q''$ nor the distributions of the respective charge densities $-\sigma'$ and $+\sigma''$ over the sphere, we can state that the total potential of the field of these charges at the special point (the center of the sphere) is zero: $\varphi_2 + \varphi_3 = 0$ (the induced charges $-Q'$ and $+Q''$ lie at equal distances from the center of the sphere, are equal in magnitude, $|-Q'| = |+Q''|$, and are opposite in sign). Hence, we need only calculate the potential φ_1 of the ring's field at O (see Figure 21.6):

$$\varphi_1 = \frac{Q}{4\pi\varepsilon_0 (l^2 + R^2)^{1/2}} . \tag{21.3}$$

This constitutes the sought potential of the sphere, $\varphi = \varphi_1$.

The terms of the problem can be made more complicated. Suppose that the charge Q on the ring is not distributed evenly. Clearly, solution (21.3) remains unchanged. Now suppose that in addition to the charge Q of the ring there are other charges contributing to the field: a point charge Q_1, and charge Q_2 carried by a thin segment, etc. It is easy to show that in all these cases the solution is reduced to calculating the field (the potential) of the new charges Q_1, Q_2, etc. at the center of the sphere. The problem has proved to be nonstandard: we are supposed to guess that the most "preferable" point is the sphere's center.

The capacitance

$$C = Q/\varphi \tag{21.4}$$

of a single conductor specifies the electric field that appears outside the conductor and on its surface (potential φ) when a charge Q is imparted to the conductor. This im-

plies that finding the capacitances of conductors is reduced to calculating the potential of this field (i.e. constitutes a basic problem of field theory).

Example 21.6. *Determine the capacitance of a single spherical conductor of radius R_1 surrounded by an adjacent concentric layer of a homogeneous insulator with a dielectric constant ε and an outer radius R_2 (Figure 21.7).*

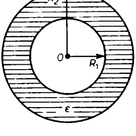

Figure 21.7

Solution. Let us impart a charge Q to the spherical conductor. Then there appears an electric field at the surface of the conductor and outside of it. If we calculate the potential of the conductor at its surface, $\varphi(R_1)$, we can use (21.4) to find the capacitance C. Calculation of the potential of the field (the field is symmetric) can be done by Gauss' method. By Gauss' law of flux,

$$D \times 4\pi r^2 = Q,$$

where $R_1 < r < R_2$. Hence, the electric field in the insulator

$$E = \frac{Q}{4\pi\varepsilon_0\varepsilon r^2}.$$

After integrating the relationship $E = -d\varphi/dr$, we arrive at the following distribution of potential in the insulator:

$$\varphi = -\int \frac{Q\,dr}{4\pi\varepsilon_0\varepsilon r^2} = \frac{Q}{4\pi\varepsilon_0\varepsilon r} + c.$$

The constant of integration can be found from the condition $\varphi(R_2) = Q/4\pi\varepsilon_0 R_2$:

$$c = \frac{Q(\varepsilon - 1)}{4\pi\varepsilon_0\varepsilon R_2}.$$

Thus, the final formula for the distribution of potential in the insulator is

$$\varphi(r) = \frac{Q}{4\pi\varepsilon_0\varepsilon}\left(\frac{1}{r} + \frac{\varepsilon - 1}{R_2}\right).$$

Employing the condition that the potential is distributed continuously, we can find the potential of the spherical conductor,

$$\varphi(R_1) = \frac{Q}{4\pi\varepsilon_0\varepsilon}\left(\frac{1}{R_1} + \frac{\varepsilon - 1}{R_2}\right),$$

and its capacitance (21.4),

$$C = \frac{Q}{\varphi(R_1)} = \frac{4\pi\varepsilon_0\varepsilon R_1}{1 + (\varepsilon - 1) R_1/R_2}.$$

Example 21.7. *The gap between the plates of a plane-parallel capacitor is filled with an isotropic insulator whose dielectric constant varies in the direction perpendicular to the plates according to the linear law from ε_1 to ε_2, with $\varepsilon_2 > \varepsilon_1$. The area of each plate is A, and the distance between the plates is d (Figure 21.8). Determine the capacitance of the capacitor.*

Solution. We point the X axis upward and place the origin on the lower plate (see Figure 21.8). Since ε varies according to the linear law, we can write

$$\varepsilon = a + bx, \tag{21.5}$$

where the constants a and b are determined from the boundary conditions ($\varepsilon = \varepsilon_1$ at $x = 0$ and $\varepsilon = \varepsilon_2$ at $x = d$):

$$a = \varepsilon_1, \quad b = (\varepsilon_2 - \varepsilon_1)/d. \tag{21.6}$$

Thus,

$$\varepsilon = \varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} x.$$

We impart a charge Q to the lower plate of the capacitor and use Gauss' method to calculate the strength of the field generated by this charge:

$$E = \frac{Q}{\varepsilon_0 \varepsilon A} = \frac{Q}{\varepsilon_0 (a + bx) A}.$$

After integrating the relationship $E = \mathrm{d}\varphi/\mathrm{d}x$, or $\mathrm{d}\varphi = Q\,\mathrm{d}x/\varepsilon_0 A\,(a + bx)$, we can determine the potential difference $\Delta\varphi$ between the plates:

$$\Delta\varphi = \int_0^d \frac{Q\,\mathrm{d}x}{\varepsilon_0 A\,(a + bx)} = \frac{Q}{\varepsilon_0 A b} \ln\left(1 + \frac{bd}{a}\right).$$

Hence, the sought capacitance is given by the formula

$$C=\frac{Q}{\Delta\varphi}=\frac{\varepsilon_0 Ab}{\ln(1+bd/a)}=\frac{\varepsilon_0 A(\varepsilon_2-\varepsilon_1)}{d\ln(\varepsilon_2/\varepsilon_1)}.$$

This problem can be generalized by assuming that the dielectric constant varies according to an arbitrary law $\varepsilon=f(x)$, with $f(x)$ an arbitrary function of x, say $f(x)=x^n$. It is easy to demonstrate that such problems can be solved by the same method.

Example 21.8. *Determine the capacitance of a spherical capacitor with plate radii R_1 and R_2 and $R_2>R_1$; the space between the plates is filled with an isotropic insulator whose dielectric constant varies according to the law $\varepsilon=a/r^2$, where a is a constant, and r the distance from the capacitor's center (Figure 21.9).*

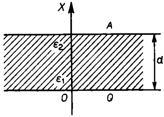

Figure 21.8

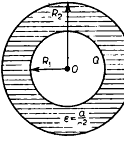

Figure 21.9

Solution. We impart a charge Q to the inner plate of the capacitor and use Gauss' method to calculate the field strength inside the insulator,

$$E=\frac{Q}{4\pi\varepsilon_0\varepsilon r^2}=\frac{Q}{4\pi\varepsilon_0 a},$$

and the potential difference between the plates,

$$\Delta\varphi=\int_{R_1}^{R_2}\frac{Q\,dr}{4\pi\varepsilon_0 a}=\frac{Q}{4\pi\varepsilon_0 a}(R_2-R_1)$$

Hence, the capacitance of such a spherical capacitor is given by the formula

$$C=\frac{4\pi\varepsilon_0 a}{R_2-R_1}.$$

To conclude this section we consider one more problem.

Example 21.9. *Determine the capacitance of a section of unit length of a two-wire line.*

Solution. The formulation of the problem is incomplete. Let us idealize the problem. We assume that the linear charge density (charge per unit length) on one wire is

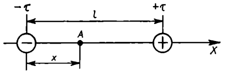

Figure 21.10

$-\tau$ and on the other, $+\tau$. We also assume that all other bodies are so far from the line that their effect on the electric field in the space between the wires can be ignored. Finally, we assume that the wires have the same radius and that $r \ll l$, where l is the distance between the wires. Thus, the physical system consists of three objects: two infinitely long thin, straight parallel wires uniformly charged with linear charge densities $-\tau$ and $+\tau$ and the electric field generated by these charges. We wish to find the capacitance of a segment of unit length of such a system.

This problem is linked to the basic problem of field theory. Let us calculate the field strength between the wires at an arbitrary point A that is positioned at a distance x from the left wire (Figure 21.10). Employing the superposition principle and the formula for the strength of the field generated by an infinitely long straight, uniformly charged string, we get

$$E=\frac{\tau}{2\pi\varepsilon_0 x}+\frac{\tau}{2\pi\varepsilon_0 (l-x)}.$$

Allowing for the relationship between field strength and potential, we get

$$\varphi = -\int E\,dx = -\frac{\tau}{2\pi\varepsilon_0}[\ln x - \ln(l-x)] + c,$$

where c is an arbitrary constant. This gives us the potentials of the left and right wires:

$$\varphi_1 = -\frac{\tau}{2\pi\varepsilon_0}[\ln r - \ln(l-r)] + c,$$

$$\varphi_2 = -\frac{\tau}{2\pi\varepsilon_0}[\ln(l-r) - \ln(r)] + c.$$

Next we find the potential difference between the wires:

$$\Delta\varphi = \varphi_1 - \varphi_2 = \frac{\tau}{\pi\varepsilon_0}\ln\frac{l-r}{r}.$$

Since $r \ll l$ by hypothesis, we have

$$\Delta\varphi = \frac{\tau}{\pi\varepsilon_0}\ln\frac{l}{r}$$

Employing relationship (21.4), we can determine the capacitance of a section of unit length of a two-wire line:

$$C = \frac{\tau}{\Delta\varphi} = \frac{\pi\varepsilon_0}{\ln(l/r)}.$$

22. Direct Current

The basic problem of direct-current theory deals with the calculation (or design) of an electric circuit. In general this problem can be formulated as follows: *given an arbitrary electric circuit and some of its parameters (emf's, resistances, etc.), find some other (unknown) quantities (currents, work, power, quantity of heat, etc.)*. Note that one should assume current I to be the most important, or fundamental, quantity in direct-current phenomena. Knowing (or finding) this quantity, we can find practically any other quantities (work, power, amount of heat, energy, parameters of the magnetic field, etc.) characterizing a particular phenomenon. Therefore, the basic problem of direct-current theory consists of finding the currents. Such a formulation of the problem is too general, and we, therefore, break it down into more concrete and narrow types.

(1) An electric circuit contains only one current source.

(2) An electric circuit contains several identical current sources.

(3) An electric circuit contains several different current sources.

Problems of the first type are solved by consistently applying Ohm's law for a closed circuit, Ohm's law for a uniform circuit element, and sometimes applying Kirchhoff's first law. If the problem is formulated correctly, the system of equations that arises from these laws is closed and, hence, the problem can be considered physically solved.

Problems of the second type can easily be reduced to those of the first if by using the rules of connecting identical current sources into batteries one finds the resulting emf of the circuit, $\mathscr{E}_0$, and by applying the rules for connecting resistances one finds the resulting internal resistance of the battery, r_0.

Problems of the first and second types are solved primarily in secondary school and will not be considered here.

Problems of the third type are the most general and cannot be reduced to problems of the first and second types. They are solved by applying laws that differ essentially from Ohm's laws for a uniform circuit element and a closed circuit. The latter cannot be applied because in the majority of such problems we cannot determine the resulting emf $\mathscr{E}_0$.

Several methods exist for solving problems of the third type. The most widespread is based on the application of Kirchhoff's laws. We will examine the essence of this method by considering a concrete example.

Example 22.1. *Determine the current flowing through a cell with an emf* $\mathscr{E}_2$ *if* $\mathscr{E}_1 = 1$ V, $\mathscr{E}_2 = 2$ V, $\mathscr{E}_3 = 3$ V, $r_1 = 1\Omega$, $r_2 = 0.5\Omega$, $r_3 = 1/3\Omega$, $R_4 = 1\Omega$, *and* $R_5 = 1/3\Omega$ *(Figure 22.1).*

Solution. The physical system consists of an electric circuit containing several different current sources. It is impossible to find the resulting emf and, therefore, we cannot apply Ohm's law for a closed circuit. In this case

the electric circuit can be calculated by applying Kirchhoff's laws.

First we must select (arbitrarily) the directions of currents in the branches. We select these directions as shown in Figure 22.1. If a direction is chosen incorrectly, the respective current will emerge as negative in the final

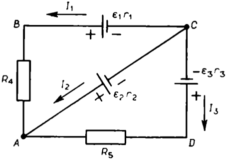

Figure 22.1

solution, and if it has been chosen correctly, the current will emerge as positive.

Let us apply Kirchhoff's first law. This law (or rule) is valid for the nodes in a circuit. The given circuit has two nodes, points A and C. For node A Kirchhoff's first law yields

$$I_1 + I_2 + I_3 = 0. \qquad (22.1)$$

For node C Kirchhoff's first law gives nothing new-

Now let us apply Kirchhoff's second law, which is valid only for closed loops. There are three loops in the given circuit: $ABCA$, $ACDA$, and $ABCDA$. We consider loop $ABCA$. It contains two emf's ($\mathscr{E}_1$ and $\mathscr{E}_2$), three resistors (r_1, r_2, and R_4), and two currents (I_1 and I_2). To apply Kirchhoff's second law we must select (arbitrarily) a sense of traversal of a loop that is positive by convention. This is necessary to determine the signs of the emf's and currents. If the directions of the emf's and currents coincide with the sense of traversal of a particular loop, the emf's and currents are considered positive; and if the directions do not coincide, they are considered negative.

We select the counterclockwise sense of traversal of loop $ABCA$ as positive. The emf $\mathscr{E}_1$ is directed counterclockwise, so it is positive; the emf $\mathscr{E}_2$ is directed clockwise (i.e. opposite to that of traversal of the loop), and so it will enter into the equation expressing Kirchhoff's second law with a "minus". Current I_1 passes through resistors r_1 and R_4 and its direction coincides with the sense of traversal of this loop. Current I_2 passes through resistor r_2 and is directed opposite to the sense of traversal. Hence, current I_1 is positive and current I_2 negative. By Kirchhoff's second law as applied to loop $ABCA$,

$$\mathscr{E}_1-\mathscr{E}_2=I_1(r_1+R_4)-I_2r_2. \qquad (22.2)$$

If we were to take the clockwise sense of traversal of the same loop as positive, Kirchhoff's second law would yield

$$-\mathscr{E}_1+\mathscr{E}_2=-I_1(r_1+R_4)+I_2r_2,$$

which is simply Eq. (22.2) multiplied by -1. Obviously, the two equations are equivalent to each other. Thus, the essence of Kirchhoff's second law does not depend on the arbitrarily chosen sense of traversal of a loop.

Let us now turn to loop $ACDA$. The counterclockwise senso of traversal of this loop will be chosen as positive. Applying Kirchhoff's second law, we obtain

$$\mathscr{E}_2-\mathscr{E}_3=I_2r_2-I_3(r_3+R_5). \qquad (22.3)$$

Equations (22.1)-(22.3) constitute a closed system. Hence, physically the problem can be considered solved. Solving the system of equation, we find that

$$I_1=-\frac{5}{8}\text{ A},\quad I_2=-\frac{1}{2}\text{ A},\quad I_3=\frac{8}{9}\text{ A}.$$

We see that currents I_1 and I_2 are negative. This means that by accident the directions of the currents I_1 and I_2 were chosen wrongly. Current I_3 is positive, which means that its direction was chosen correctly.

Example 22.2. *A cylindrical air capacitor with an inner radius R_1 and an outer radius R_2 has been charged to a potential difference $\Delta\varphi_0$ (Figure 22.2). The space between the plates is filled with a low-conducting material with q*

resistivity ρ. Find the leakage current if the height (or length) of the capacitor is l.

Solution. The physical system consists of the section of the electric circuit in which the reason for the directional motion of the free charges in the low-conducting medium is the electrostatic field existing in the space between the plates. We consider the potential difference $\Delta\varphi_0$ of this field as constant. Since the circuit element is uniform (i.e. there are no emf's in it), the current can be found by applying Ohm's law for a uniform circuit element,

$$I = \Delta\varphi_0/R, \qquad (22.4)$$

if we know the total resistance R of this element. This quantity, R, can be found via the DI method. The elementary resistance of a thin-walled cylindrical layer of thickness (length) dr and radius r (see Figure 22.2) is given by the formula

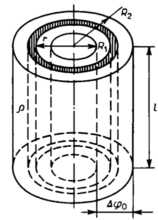

Figure 22.2

$$\mathrm{d}R = \rho\,\frac{\mathrm{d}r}{2\pi r l}.$$

Integration yields the value of the total resistance of the given element:

$$R = \int_{R_1}^{R_2} \rho\,\frac{\mathrm{d}r}{2\pi l r} = \frac{\rho}{2\pi l}\ln\frac{R_2}{R_1}. \qquad (22.5)$$

Hence

$$I_0 = \frac{2\pi l \Delta\varphi_0}{\rho \ln (R_2/R_1)}. \qquad (22.6)$$

Is solution (22.6) always valid? We can also pose a number of other questions. We have said that after determining the basic quantity, current, it is easy to deter-

mine all other parameters of a circuit and the processes that occur during the passage of current. For one, from Joule's law,

$$Q = I^2Rt, \tag{22.7}$$

we can find the amount of heat Q liberated by a circuit element during time t; or the law

$$q = It \tag{22.8}$$

can be used to find the amount of electricity that has passed through a circuit element during time t.

Let us find the time interval t_0 during which the charge initially on the capacitor, $q_0 = C\ \Delta\varphi_0$, will pass through the circuit element, with

$$C = \frac{2\pi\varepsilon_0\varepsilon l}{\ln(R_2/R_1)} \tag{22.9}$$

the capacitor's capacitance. The solution

$$t_0 = \frac{q_0}{I} = \frac{C\Delta\varphi_0\rho\ln(R_2/R_1)}{2\pi l\Delta\varphi_0} = \varepsilon_0\varepsilon\rho \tag{22.10}$$

is formal and essentially incorrect. Indeed, solution (22.10) is valid only if the current is constant. From (22.4) and (22.6) it follows that condition I = const is met only if R = const and $\Delta\varphi_0$ = const. But condition $\Delta\varphi_0$ = const means that at C = const the charge on the plates of the capacitor must remain constant, which contradicts the terms of the problem. The charge on the plates decreases, which means that the potential difference $\Delta\varphi$ also decreases. This implies that, strictly speaking, the leakage current does not remain constant, as it should according to (22.6). Solution (22.6) is valid if $\Delta\varphi$ = const, which, strictly speaking, is never the case.

Obviously, the way out of this situation is as follows: we consider the potential difference $\Delta\varphi$ as changing so slowly in the given conditions that these changes can be ignored and it can be assumed that $\Delta\varphi$ = const. This condition corresponds to the notion of a "low-conducting medium" in Example 22.2. Now we make the problem more complicated by assuming that in reality the potential difference $\Delta\varphi$ is not constant.

Example 22.3. *Using the terms of Example 22.2, find the law of variation of the leakage current with time.*

Solution. Suppose that the current changes so slowly that at every fixed moment in time Ohm's law (22.4) holds. Then

$$-\frac{dq}{dt}=\frac{\Delta\varphi}{R}, \tag{22.11}$$

where $I = -dq/dt$ and $\Delta\varphi = q/C$ are the instantaneous values of current and potential difference. Thus, we have arrived at the differential equation

$$-\frac{dq}{dt}=\frac{q}{CR} \tag{22.12}$$

for the unknown function $q = q(t)$, which is the charge q on the plates of the capacitor at any moment t in time. Separating variables and integrating, we get

$$\ln q = -\frac{1}{RC}t + c_1.$$

Allowing for the initial conditions ($q = q_0 = C\,\Delta\varphi_0$ at $t = 0$), we can find the constant: $c_1 = \ln q_0$. Thus,

$$q = q_0 e^{-t/RC}. \tag{22.13}$$

We will call the constant $\tau = RC$ the *relaxation time*, or the *relaxation constant*. It is easy to see that τ is the time it takes the initial charge q_0 to diminish by a factor of $e \approx 2.78$ . . . . In our case,

$$\tau = \frac{\rho \ln(R_2/R_1)}{2\pi l}\,\frac{2\pi\varepsilon_0 \varepsilon l}{\ln(R_2/R_1)} = \varepsilon_0\varepsilon\rho, \tag{22.14}$$

which coincides with (22.10). Thus, t_0 is not the time that it takes the entire charge q_0 to flow from one plate to the other, but only the relaxation time. From (22.13) we see that it takes an infinitely long time for the charge to flow from one plate to the other ($q = 0$ at $t = \infty$).

From (22.13) we can also arrive at the laws of variation of other quantities, the potential difference and the cur-

rent:

$$\Delta\varphi = \frac{q}{C} = \frac{q_0}{C} e^{-t/RC} = \Delta\varphi_0 e^{-t/RC}, \tag{22.15}$$

$$I = -\frac{dq}{dt} = \frac{q_0}{RC} e^{-t/RC} = \frac{\Delta\varphi_0}{R} e^{-t/RC} = I_0 e^{-t/RC}. \tag{22.16}$$

Thus, formula (22.6) gives only the initial value of the leakage current.

Now let us clarify the meaning of the concept of a low-conducting medium. This is a medium in relation to which we can ignore the variations in the instantaneous values of current (22.16), potential difference (22.15), charge (22.13), and other quantities. These variations can be ignored if the relaxation time τ is relatively large. Hence, a low-conducting medium is one in which the relaxation time τ is relatively large.

Let us estimate the relaxation times of some of the materials in our case. We start with paraffin ($\varepsilon = 2$, $\rho = 3 \times 10^{16}\ \Omega\cdot\text{m}$). Then (22.14) yields

$$\tau = \varepsilon_0 \varepsilon \rho, \quad \tau \approx 5.3 \times 10^5 \text{ s} \approx 6.1 \text{ days}.$$

Let us assume, for the sake of convenience, that the duration of observation is $\tau_0 \approx 1$ s. Thus, paraffin can be considered a low-conducting medium with a very close approximation.

If we use formula (22.14) to estimate the relaxation times of two types of quartz with $\varepsilon_1 = 4.4$, $\rho_1 = 3 \times 10^{14}\ \Omega\cdot\text{m}$ and $\varepsilon_2 = 4.7$, $\rho_2 = 1 \times 10^{12}\ \Omega\cdot\text{m}$ and of marble with $\varepsilon_3 = 8.3$, $\rho_3 = 1 \times 10^8\ \Omega\cdot\text{m}$, we obtain 3.25 hr, 42 s, and 7×10^{-3} s, respectively. Hence, while both types of quartz can be considered low-conducting materials, marble cannot. Of course, if the characteristic duration of observation is $\tau_0 = 10^{-6}$ s, we can even consider marble a low-conducting medium.

Chapter 7

THE MAGNETIC FIELD

23. The Magnetic Field in a Vacuum

When studying magnetic fields, we must include in the physical system the sources of magnetic fields and their fields.

> The basic problem of the theory of magnetic field (as well as the theories of gravitation and electric field) consists of calculating the characteristics of the magnetic field of an arbitrary system of currents and moving electric charges,

which is equivalent to determining the magnetic induction **B** at an arbitrary point in the field. This problem is solved by applying the Biot-Savart law in differential form

$$d\mathbf{B} = \frac{\mu_0 I \, d\mathbf{l} \times \mathbf{r}}{4\pi r^3}, \quad (23.1)$$

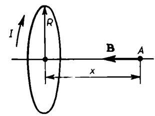

Figure 23.1

the superposition principle, and the DI method (see Section 6). A theorem often employed in this connection is concerned with the circulation of vector **B**,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \sum I, \quad (23.2)$$

especially in cases where (23.1) is invalid.

It is advisable first to solve several elementary problems involving two widespread sources of magnetic field (the results will be given without calculations): the magnetic induction of a circular current I of radius R at a point A on the axis at a distance x from the plane of the current (Figure 23.1),

$$B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}}, \quad (23.3)$$

and the magnetic induction of a section of straight wire carrying a current I at a point A lying at a distance r_0

from the wire (Figure 23.2),

$$B = \frac{\mu_0 I}{4\pi r_0}(\cos\alpha_1 - \cos\alpha_2). \qquad (23.4)$$

Then we can formulate and solve literally dozens of problems involving the calculation of magnetic fields generated by various combinations of the above sources: squares, triangles, rectangles, trapezoids, figures formed by combining circles, half-lines, segments, etc. All these problems can be solved by the superposition method. What is most essential is to allow for the vector nature of the superposition principle.

Example 23.1. *Find the magnitude of the magnetic induction* **B** *of a magnetic field generated by a system of*

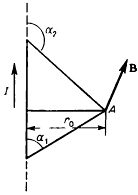

Figure 23.2

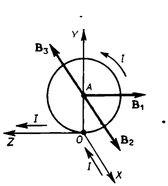

Figure 23.3

thin conductors (Figure 23.3) along which a current I is flowing at a point A $\{0, R, 0\}$ that is the center of a circular conductor of radius R.

Solution. The magnetic field is generated by three sources: the XO half-line (the positive half of the X axis), the circular conductor of radius R centered at point A $\{0, R, 0\}$ and lying in the ZOY plane, and the half-line OZ (the positive half of the Z axis). The current in all three conductors is the same, I. The magnetic induction $\mathbf{B}_1$ of

the magnetic field generated by the current flowing in the XO conductor lies in the ZOY plane and points in the direction opposite to that of the Z axis, the magnetic induction $\mathbf{B}_2$ of the magnetic field generated by the current flowing in the circular conductor lies in the XOY plane and points in the direction opposite to that of the X axis, and the magnetic induction $\mathbf{B}_3$ of the magnetic field generated by the current flowing in the OZ conductor lies in the same XOY plane but points in the direction opposite to that of $\mathbf{B}_3$ (Figure 23.3). We use (23.4) to find the magnitudes of $\mathbf{B}_1$ and $\mathbf{B}_3$,

$$B_1 = B_3 = \frac{\mu_0 I}{4\pi R},$$

and (23.3) to find the magnitude of $\mathbf{B}_2$,

$$B_2 = \frac{\mu_0 I}{2R}.$$

According to the superposition principle,

$$B = \sqrt{B_1^2 + (B_2 - B_3)^2} = \frac{\mu_0 I}{4\pi R} \sqrt{2(2\pi^2 - 2\pi + 1)}.$$

Example 23.2. *A current of density j flows along an infinitely long solid cylindrical conductor of radius R. Calculate the magnetic induction inside and outside the conductor.*

Solution. Since the conductor is not thin, the Biot-Savart law (23.1) and its corollary (23.4) cannot be employed. Let us use the circulation theorem (23.2). We consider a point A_1 lying at a distance r_1 from the conductor's axis (Figure 23.4). We draw a circle of radius r_1 centered at point O on the conductor's axis. In view of the symmetry of the problem, the magnitude of $\mathbf{B}_1$ is the same at each point of the circle. The sum of the currents ΣI encompassed by this loop (the circle) is equal to $j\pi r_1^2$. Thus, by the circulation theorem (23.2),

$$B_1 \times 2\pi r_1 = \mu_0 j \pi r_1^2.$$

This yields the following formula for the magnitude of the magnetic induction at point A_1:

$$B_1 = \frac{1}{2} \mu_0 j r_1. \qquad (23.5)$$

Let us take a point A_2 lying at a distance $r_2 > R$ from the conductor's axis (see Figure 23.4). Applying the circulation theorem, we get

$$B_2 \times 2\pi r_2 = \mu_0 j \pi R^2.$$

Hence, the magnitude of the magnetic induction outside the field is given by the formula

$$B_2 = \frac{\mu_0 j R^2}{2r_2}. \qquad (23.6)$$

The diagram that demonstrates the behavior of the magnetic induction of a solid cylindrical conductor is given in Figure 23.5.

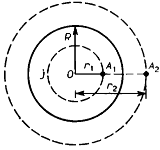

Figure 23.4

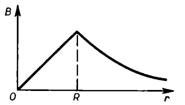

Figure 23.5

Example 23.3. *A thin band of width l has been wound into the shape of a pipe of radius R (Figure 23.6). A current I flows along the pipe and is evenly distributed throughout its width (the direction of the current is shown in Figure 23.6). Determine the magnitude of the magnetic induction at an arbitrary point on the pipe's axis.*

Solution. The conductor can be considered to be neither thin nor a current element. Hence, we cannot employ the Biot-Savart law (23.1) or its corollary (23.3). It is also difficult to employ the circulation theorem (23.2) since the magnetic field possesses no symmetry. Let us employ the DI method.

We partition the pipe into rings so narrow that each can be considered to be a thin circular conductor. We

consider one such narrow ring of width dx positioned at a distance x from an arbitrary point A_1 (Figure 23.6).

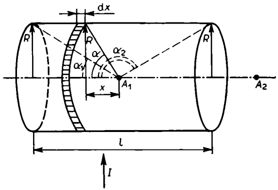

Figure 23.6

The elementary current in this narrow ring,

$$dI = \frac{dx}{l} I, \tag{23.7}$$

generates, according to A_1, a magnetic field whose elementary magnetic induction is

$$dB = \frac{\mu_0 I R^2\, dx}{2l\,(R^2 + x^2)^{3/2}}.$$

It is convenient to take the angle α at which the radius of each thin ring is seen from point A_1 as the integration variable. Since

$$x = R\cot\alpha, \quad dx = -\frac{R\, d\alpha}{\sin^2\alpha}, \quad R^2 + x^2 = \frac{R^2}{\sin^2\alpha},$$

we have

$$dB = \frac{\mu_0 I \sin\alpha\, d\alpha}{2l}.$$

Integration yields

$$B = \int_{\alpha_1}^{\alpha_2} \frac{\mu_0 I \sin\alpha\, d\alpha}{2l} = \frac{\mu_0 I}{2l}(\cos\alpha_1 - \cos\alpha_2). \tag{23.8}$$

If we introduce the concept of a current per unit length of pipe,

$$I_0 = I/l, \qquad (23.9)$$

then (23.8) assumes the form

$$B = \frac{\mu_0 I_0}{2} (\cos\alpha_1 - \cos\alpha_2). \qquad (23.10)$$

This formula is also true for a solenoid if we allow for the obvious formula $I_0 = nI_1$, where n is the number of turns of wire per unit length of solenoid, and I_1 the current in the solenoid. Hence, for a finite solenoid we have

$$B = \frac{\mu_0 n I_1}{2} (\cos\alpha_1 - \cos\alpha_2). \qquad (23.11)$$

The above formulas (23.8), (23.10), and (23.11) also hold true for a point A_2 positioned on the axis outside the solenoid (see Figure 23.6). Note that for point A_1 the angle α_2 is always obtuse, while for point A_2 it is always acute (excluding the points at the end faces). It is useful to study various particular cases; point A_1 is at the middle of the pipe or at its ends, the pipe is infinitely long, or the solenoid is infinitely long ($l \to \infty$).

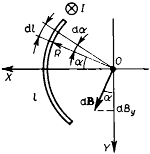

Figure 23.7

Example 23.4. *A current I flows in a long straight conductor whose cross section has the shape of a thin arc of length l and radius R (Figure 23.7). Determine the magnetic induction x of the magnetic field induced by the current at point O.*

Solution. It is easy to see that the conductor can be considered to be neither a thin straight conductor nor a current element. Hence, we cannot employ the Biot-Savart law (23.1) or its corollary (23.4). Since the magnetic field possesses no symmetry, it is doubtful that the circulation theorem (23.2) will yield any results. Let us apply the DI method.

We partition the conductor into long, straight conductors so narrow that each can be thought of as a long, thin, straight conductor. From (23.4) it follows that the magnetic induction produced by the current in an infinitely long, thin, straight conductor can be calculated via the formula

$$B = \frac{\mu_0 I}{2\pi r_0}. \tag{23.12}$$

Let us consider one such conductor whose width is dl (Figure 23.7). The elementary current in such a conductor is

$$dI = \frac{dl}{l} I,$$

and at point O it generates a magnetic field whose elementary magnetic induction is (see (23.12))

$$dB = \frac{\mu_0 \, dI}{2\pi r} = \frac{\mu_0 I \, dl}{2\pi l R}.$$

It is easy to see that the resultant vector **B** is directed along the Y axis (i.e. $B_x = 0$). The projection of vector $d\mathbf{B}$ on the Y axis is

$$dB_y = \frac{\mu_0 I \, dl}{2\pi l R} \cos\alpha.$$

For the integration variable we take angle α. Since $dl = R \, d\alpha$, we have

$$dB_y = \frac{\mu_0 I \cos\alpha \, d\alpha}{2\pi l}.$$

Integration yields

$$B_y = \int\limits_{-\alpha_0/2}^{+\alpha_0/2} \frac{\mu_0 I \cos\alpha \, d\alpha}{2\pi l} = \frac{\mu_0 I}{\pi l} \sin\frac{\alpha_0}{2}, \tag{23.13}$$

where $\alpha_0 = l/R$ is the central angle of the arc l. If $\alpha_0 = \pi$, (23.13) yields

$$B_y = \frac{\mu_0 I}{\pi^2 R}.$$

Example 23.5. *A current I flows along an infinitely long-thin, straight band of width l. Calculate the magnetic induc-*

tion of the magnetic field generated by this current at an arbitrary point O (Figure 23.8).

Solution. With point O we link a coordinate system whose axes are directed as shown in Figure 23.8. To calculate the magnetic field we employ the DI method (as in the two previous examples, neither the Biot-Savart law nor its corollary can be applied).

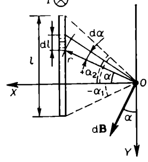

Figure 23.8

Let us partition the infinitely long band into infinitely long, narrow, straight sections each of which can be thought of as an infinitely long, thin, straight conductor. Let us consider one such section of width dl (see Figure 23.8). The elementary current flowing in this section is

$$dI = \frac{dl}{l} I,$$

and it generates at point O a magnetic field whose induction is given by the formula (see (23.12))

$$dB = \frac{\mu_0 \, dI}{2\pi r} = \frac{\mu_0 I dl}{2\pi l r}.$$

Suppose that point O is r_0 distant from the plane of the band. Then

$$r = \frac{r_0}{\cos\alpha}, \; dl = \frac{r\, d\alpha}{\cos\alpha} = \frac{r_0 \, d\alpha}{\cos^2\alpha}.$$

Thus,

$$dB = \frac{\mu_0 I \, d\alpha}{2\pi l \cos\alpha}.$$

Let us find the projections of vector $\mathbf{dB}$ on the X and Y axes:

$$dB_x = dB \sin\alpha = \frac{\mu_0 I \sin\alpha \, d\alpha}{2\pi l \cos\alpha}, \quad dB_y = dB\cos\alpha = \frac{\mu_0 I \, d\alpha}{2\pi l}.$$

Integration yields

$$B_x = \int_{-\alpha_1}^{+\alpha_2} \frac{\mu_0 I \sin\alpha \, d\alpha}{2\pi l \cos\alpha} = \frac{\mu_0 I}{2\pi l} \ln \frac{\cos\alpha_1}{\cos\alpha_2}, \qquad (23.14)$$

$$B_y = \int_{-\alpha_1}^{+\alpha_2} \frac{\mu_0 I \, d\alpha}{2\pi l} = \frac{\mu_0 I \, (\alpha_2 + \alpha_1)}{2\pi l}. \qquad (23.15)$$

Introducing the concept of a current per unit width of band, $I_0 = I/l$, we find that

$$B_x = \frac{\mu_0 I_0}{2\pi} \ln \frac{\cos\alpha_1}{\cos\alpha_2}, \qquad B_y = \frac{\mu_0 I_0}{2\pi} (\alpha_2 + \alpha_1).$$

In the case where point O is positioned symmetrically (i.e. $\alpha_1 = \alpha_2$) we have $B_x = 0$ and $B_y = \mu_0 I \alpha_1 / \pi$. For a band of infinite width (i.e. a plane) we have $B_x = 0$ and $B_y = \mu_0 I_0 / 2$ (i.e. the field generated by a plane with an evenly distributed current I_0 is homogeneous).

If the magnetic induction is known (or has been calculated by the described methods), the solution of most problems is reduced to solving the respective problems of mechanics (often by applying the DI method). The most widespread problems are those related to the behavior of a flat current-carrying loop in a magnetic field. Often one has to calculate the forces and torques acting on the loop, determine the work done in the process of moving the loop in a magnetic field, etc.

Example 23.6. *A thin conductor in the shape of a semicircle of radius R carries a current I in the direction shown in Figure 23.9. The conductor is placed in a homogeneous magnetic field with a magnetic induction $\mathbf{B} = \{0, \mathbf{B}_0, 0\}$. Determine the force acting on the conductor.*

Solution. It would be a mistake to use Ampère's law in the form $F = IlB_0$, where $l = \pi R$ is the length of the conductor, since each element of the conductor is positioned differently in relation to the magnetic field. Let us apply the DI method.

We partition the conductor into sections so small that each can be considered to be a current element. We take one such section whose length is dl. The magnitude

of the elementary force $\mathbf{dF}$ acting on this section is, by Ampère's law,

$$dF = I\, dl\, B_0 \sin\alpha. \qquad (23.16)$$

It is easy to see that all the elementary vectors $\mathbf{dF}_i$ are directed along the Z axis. Hence, vector summation is

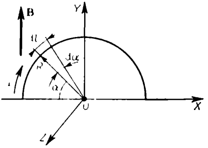

Figure 23.9

reduced to arithmetic. Since $dl = R\, d\alpha$, integration of (23.16) with respect to angle α yields

$$F = \int_0^{\pi} IRB_0 \sin\alpha\, d\alpha = 2IRB_0.$$

Many variants of this problem can be considered: the magnetic induction may be directed along the X axis or along the Z axis or at various angles to the coordinate axes. All these problems can be solved by the DI method.

Example 23.7. *A square current-carrying loop made of thin wire and having a mass $m = 10$ g can rotate without friction with respect to the vertical axis OO_1 passing through the center of the loop at right angles to two opposite sides of the loop (Figure 23.10). The loop is placed in a homogeneous magnetic field with an induction $B = 10^{-1}$ T directed at right angles to the plane of the drawing. A current $I = 2$ A is flowing in the loop. Find the period of small oscillations that the loop performs about its position of stable equilibrium.*

Solution. The physical system consists of a known (homogeneous) magnetic field, the current-carrying square loop, and the free charges moving in the material of the loop (current I). The physical phenomenon consists of the loop performing small oscillations under the forces exerted by the magnetic field on each current element. Since the magnetic induction is known, we can find these forces and their resulting torque.

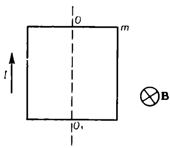

Figure 23.10

When the loop turns through a small angle α away from its position of equilibrium, a moment of Ampère forces act on it:

$$M = p_m B \sin\alpha, \tag{23.17}$$

where

$$p_m = IA = Ia^2 \tag{23.18}$$

is the magnetic moment of the loop, and a the length of the loop's side. Applying the equation of motion (14.7), we obtain

$$J\beta = M, \tag{23.19}$$

where J is the moment of inertia of the loop about the OO_1 axis, and $\beta = \ddot{\alpha}$ the angular acceleration of the loop. The loop's moment of inertia is

$$J = 2\times\frac{m}{4}\times\frac{a^2}{4} + 2\times\frac{1}{12}\times\frac{m}{4}\times a^2 = \frac{1}{6}ma^2. \tag{23.20}$$

Substituting into Eq. (23.19) the value of the moment of Ampère forces, (23.17), and the value of the moment of inertia of the loop, (23.20), we obtain the following equation:

$$\ddot{\alpha} + \frac{6IB}{m}\sin\alpha = 0.$$

Bearing in mind that $\sin\alpha \approx \alpha$ for small α's, we obtain a differential equation for the harmonic oscillations of the loop:

$$\ddot{\alpha}+\frac{6IB}{m}\alpha=0. \qquad (23.21)$$

If we compare this equation with the general equation for harmonic oscillations, we arrive at the following formula for the angular frequency of the loop's oscillations,

$$\omega_0=\sqrt{6IB/m},$$

and the period of oscillations,

$$T_0-2\pi\sqrt{m/6IB}, \quad T_0\approx 0.57 \text{ s}.$$

As is known, when a flat current-carrying loop is moved in a magnetic field, the following amount of work is performed:

$$W=I\Delta\Phi, \qquad (23.22)$$

where $\Delta\Phi$ is the variation of the magnetic flux passing through the area limited by the loop. If a point magnetic dipole (a flat loop carrying a current I and having sufficiently small dimensions) whose magnetic-moment vector is

$$p_m=IA\mathbf{n} \qquad (23.23)$$

is parallel to the magnetic induction $\mathbf{B}$, calculating the work amounts to calculating the magnetic field:

$$W=I\Delta\Phi=I(\Phi_1-\Phi_2)=\frac{p_m}{A}(B_1-B_2)A=p_m(B_1-B_2). \qquad (23.24)$$

Example 23.8. *Assuming that the terms of Example 23.3 remain valid, we place a point magnetic dipole with magnetic moment* $\mathbf{p}_m$ *at point* A_1 *in the middle of the pipe's axis (Figure 23.11). The dipole is then moved from point* A_1 *to point* A_2 *along the axis in such a manner that vector* $\mathbf{p}_m$ *remains parallel to vector* $\mathbf{B}$. *Find the work done in moving the dipole.*

Solution. Equation (23.24) shows that to solve the problem it is sufficient to calculate the induction B_1 of the magnetic field at point A_1 and induction B_2 at point A_2.

According to (23.8), we have

$$B_1 = \frac{\mu_0 I}{2} \frac{1}{\sqrt{R^2 + l^2/4}}, \quad B_2 = \frac{\mu_0 I}{2} \frac{1}{\sqrt{R^2 + l^2}}.$$

Substituting these values into (23.24), we find that

$$W = \frac{\mu_0 p_m I}{2} \left(\frac{1}{\sqrt{R^2 + l^2/4}} - \frac{1}{\sqrt{R^2 + l^2}} \right).$$

If the magnetic dipole that is moved in a magnetic field is not point-like but an ordinary flat loop carrying

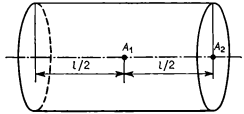

Figure 23.11

a current I, the DI method is often used to calculate the magnetic flux.

Example 23.9. *An infinitely long, straight wire carrying a current $I_1 = 5$ A lies in the plane of a rectangular loop carrying a current $I_2 = 3$ A. The two are positioned in such a manner that one side of the rectangular loop whose length is $l = 1$ m is parallel to the straight wire and is $r = 0.1\ b$ distant from it, with b the length of the other side of the rectangular loop (Figure 23.12). Determine the work that must be done so that the loop is turned through an angle $\alpha = 90°$ about the OO_1 axis, which is parallel to the straight wire and passes through the middle of the opposite sides of the loop whose length is b.*

Solution. It is easy to see that when the loop is in its final position, the magnetic flux passing through it is zero: $\Phi_2 = 0$. Thus, we need only to calculate the magnetic flux Φ_1 passing through the loop in the initial position. Since the field generated by a current I_1 flowing along an

infinitely long, straight wire,

$$B_1 = \frac{\mu_0 I_1}{2\pi r} \tag{23.25}$$

(see (23.4)), is nonhomogeneous, the solution $\Phi_1 = B_1 A$, with $A = lb$ the area of the loop, is incorrect. We, therefore, apply the DI method.

We partition the area encompassed by the loop into strips so narrow that within each strip the magnetic

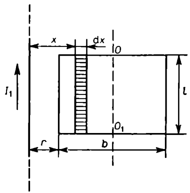

Figure 23.12

field can be assumed to be homogeneous. We take one such strip of width dx (see Figure 23.12) that is x distant from the straight wire carrying current I_1. The elementary magnetic flux passing through this strip is

$$d\Phi = B\,dA = \frac{\mu_0 I_1}{2\pi x}\, l\,dx. \tag{23.26}$$

After integrating with respect to x, we find the sought magnetic flux:

$$\Phi_1 = \int_{0.1b}^{0.1b+b} \frac{\mu_0 I_1 l}{2\pi x}\, dx = \frac{\mu_0 I_1 l}{2\pi} \ln 11.$$

Thus,

$$W = I_2 \Delta\Phi = I_2 \Phi_1 = \frac{\mu_0 I_1 I_2 l}{2\pi} \ln 11.$$

Substitution of numerical values yields $W \approx 7.1 \times 10^{-6}$ J.

24. The Magnetic Field in Matter

When considering the magnetic field in a magnetic substance, we introduce two physical quantities in addition to the magnetic induction **B**: the *magnetization* **J** (the magnetic moment per unit volume of the substance) and the *magnetic field strength*

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{J}. \tag{24.1}$$

For a homogeneous and isotropic magnetic substance,

$$\mathbf{B} = \mu_0 \mu \mathbf{H}. \tag{24.2}$$

The relative permeability of a ferromagnetic substance, μ, *is a nonlinear function of the magnetic field strength* H. Hence, when solving problems involving ferromagnetic substances one often uses empirical B vs. H curves. Figure 24.1 shows such curves for iron, steel, and pig iron.

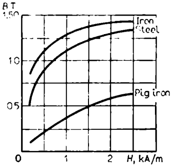

Figure 24.1

Finding the magnetic induction vector **B** constitutes the basic problem of the theory of magnetic substances. A common procedure is to employ the magnetic circulation theorem,

$$\oint \mathbf{H} \cdot d\mathbf{l} = \sum I, \tag{24.3}$$

the B vs. H curves of the type shown in Figure 24.1, and the fact that at the boundary between two different magnetic substances the normal component of vector **B** varies continuously:

$$B_{1n} = B_{2n}. \tag{24.4}$$

Example 24.1. *A closed toroid with an iron core has* $N = 400$ *turns of thin wire in a single layer. The mean diameter of the toroid is* $d = 25$ cm. *Determine the magnetic*

field strength and the magnetic induction inside the toroid, the permeability μ *of the iron, and the magnetization* J *for two values of the current flowing in the wire:* $I_1 = 0.5$ A and $I_2 = 5$ A.

Solution. Applying the theorem on the circulation of H (24.3) along the circumference of diameter d (the

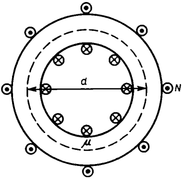

Figure 24.2

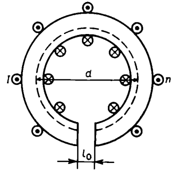

Figure 24.3

median line of the toroid; Figure 24.2),

$$H \times \pi d = IN,$$

we find the magnetic field strength inside the toroid:

$$H = IN/\pi d.$$

Substituting the values of the current, we get

$$H_1 = 255 \text{ A/m}, \quad H_2 = 2550 \text{ A/m}.$$

Next, using the curve for iron in Figure 24.1, we determine the respective values of magnetic induction:

$$B_1 = 0.9 \text{ T}, \quad B_2 = 1.45 \text{ T}.$$

We can now use Eq. (24.2) to find the values of the relative permeability $\mu = B/\mu_0 H$ of the iron core:

$$\mu_1 \approx 2.8 \times 10^3, \quad \mu_2 \approx 4.5 \times 10^2.$$

Finally, using Eq. (24.1), we can find the magnetization $J = B/\mu_0 - H$:

$$J_1 \approx 7.1 \times 10^5 \text{ A/m}, \quad J_2 \approx 1.1 \times 10^6 \text{ A/m}.$$

Analysis of the above data shows that only the magnetic field strength inside the ferromagnetic substance (iron) is directly proportional to the current flowing in the wire, while the other quantities, namely, induction B, permeability μ, and magnetization J, are nonlinear functions of H and, hence, of the current I.

Example 24.2. *The winding of a toroid with an iron core containing a vacuum gap that is* $l_0 = 3$ mm *wide has* $n = 1000$ *turns per meter. The mean diameter of the toroid is* $d = 30$ cm. *What must the current I flowing in the winding be if the magnetic induction* B_0 *in the gap is to be* 1 T *(Figure 24.3)*?

Solution. Employing the theorem on the circulation of vector H (24.3), we find that

$$H\pi d + H_0 l_0 = I\pi n d, \tag{24.5}$$

where H is the magnetic field strength in the core, and H_0 the magnetic field strength in the gap. Since the relative permeability μ of a vaccuum is unity, we can use (24.2) and find the magnetic field strength in the gap:

$$H_0 = \frac{B_0}{\mu_0}, \quad H_0 = \frac{10^7}{4\pi} \text{ A/m}. \tag{24.6}$$

Because the vacuum gap is narrow, we assume that the radial components of the magnetic induction in gap and in core are zero. Then, allowing for (24.4), we conclude that induction B in the core is equal in absolute value to B_0. Using the appropriate curve in Figure 24.1, we can find the magnetic field strength in the core: $H = 7 \times 10^2$ A/m. Thus, Eq. (24.5) yields

$$I = \frac{H}{n} + \frac{B_0 l_0}{\mu_0 \pi n d}, \quad I \approx 3.2 \text{ A}. \tag{24.7}$$

Example 24.3. *Let us change the terms of Example 24.2. Suppose that the current flowing in the winding of the toroid is* $I = 3.2$ A. *Find the magnetic induction* B_g *in the gap. All other conditions remain unaltered.*

Solution. At first glance we seem to have formulated a problem that is the reverse of the Problem 24.2 and the solution can be obtained by employing formulas (24.5)

and (24.6). This system of equations contains three unknowns, H, H_0, and B_0, and although these are connected through the appropriate curve in Figure 24.1, we cannot employ the curve directly. Nevertheless, the problem

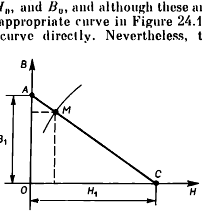

Figure 24.4

is solved via a diagram. Since $B = B_0$, from (24.7) we can find the relationship between B, H, and I:

$$B = \frac{I\pi\mu_0 nd}{l_0} - \frac{\mu_0 \pi d}{l_0} H. \tag{24.8}$$

For a given I, Eq. (24.8) specifies a linear B vs. H dependence for different cores. For a given core (made of steel), the values of B and H in (24.8) must satisfy the relationship specified by the appropriate curve in Figure 24.1. Hence, the sought values of B and H are the parameters of the point $M = \{B, H\}$ where the curve in Figure 24.1 corresponding to steel intersects the straight line reflecting Eq. (24.8) (Figure 24.4). In Figure 24.4 the straight line (24.8) intersects the coordinate axes at points $A = \{0, B_1\}$ and $C = \{H_1, 0\}$, with

$$B_1 = \frac{I\pi\mu_0 nd}{l_0}, \quad B_1 \approx 1.26\ \text{T},$$

and

$$H_1 = In, \quad H_1 = 3.2 \times 10^3\ \text{A/m}.$$

The reader can easily see that the coordinates of point M are $B = B_0 = 1$ T and $H = 700$ A/m, which corresponds to the data of Example 24.2.

Chapter 8

THE ELECTROMAGNETIC FIELD

25. Electromagnetic Induction and Self-Induction

The basic law governing the electromagnetic induction phenomenon is Faraday's law

$$\mathscr{E}_i = -\frac{d\Phi}{dt}. \qquad (25.1)$$

Hence,

> the problem of finding the induction emf $\mathscr{E}_i$ constitutes the basic problem in the electromagnetic induction theory.

When performing the stage of physical analysis one must thoroughly investigate the causes of the changes in the magnetic flux Φ and how this quantity actually varies. Then one must determine the magnetic flux passing through the surface encompassed by the loop as a function of time t, that is, $\Phi = \Phi(t)$. Now Faraday's law (25.1) can be used to find the induction emf.

Example 25.1. *A flat square loop with a side $a = 20$ cm is placed in a magnetic field whose induction $\mathbf{B} = (\alpha + \beta t^2)\mathbf{i}$, where $\alpha = 10^{-1}$ T, $\beta = 10^{-2}$ T/s^2, and $\mathbf{i}$ is the unit vector pointing along the X axis; the plane of the loop is at right angles to $\mathbf{B}$. Find the induction emf generated in the loop at time $t = 5$s.*

Solution. The physical system consists of the magnetic field varying with time, the loop placed in this field, and the induction current generated by this field in the loop. We want to know the induction emf. The reason why the magnetic flux passing through the loop varies is the variation with time of the magnetic induction vector. Let us find the magnetic flux.

Since the magnetic field is homogeneous and the loop is flat, we have

$$\Phi = \mathbf{B}\cdot\mathbf{A} = (\alpha + \beta t^2)\, a^2. \qquad (25.2)$$

This gives the following formula for the induction emf:

$$|\mathscr{E}_i| = \left| -\frac{d\Phi}{dt} \right| = 2\beta a^2 t, \quad |\mathscr{E}_i| = 4 \times 10^{-3}\,\text{V}. \qquad (25.3)$$

As we have repeatedly done in the past, we can now simplify the problem or make it more complicated, thus formulating dozens of new similar problems. But what does "similar" mean in this context? Here we consider ways of constructing clusters of problems and introduce the concept of the generalized problem of a cluster. The word "similar" means that in all problems of a specific type the underlying phenomenon is the same (in our case it is electromagnetic induction), the magnetic field is homogeneous and varies in time, and the loop is flat and is positioned at right angles to **B**. To solve all such problems (the cluster) we can employ Eqs. (25.2) and (25.3).

We can now formulate the generalized problem of the first "cluster": *a flat loop encompassing an area A is placed in a magnetic field whose induction varies according to the law* $\mathbf{B} = f(t)\,\mathbf{i}$, *where* $f(t)$ *is an arbitrary (differentiable) function of time t, in such a manner that the plane of the loop is at right angles to* **B**; *we wish to determine the induction emf in the loop at an arbitrary moment in time.*

The generalized problem of the first cluster can be solved by the same equations (25.2) and (25.3) but in generalized form:

$$\Phi = \mathbf{B}\cdot\mathbf{A} = f(t)\,\mathbf{i}\cdot\mathbf{A}, \tag{25.4}$$

$$|\mathscr{E}_i| = \left| -\frac{d\Phi}{dt} \right| = f'(t)\,A. \tag{25.5}$$

Now the solution of any specific problem belonging to this cluster can be solved directly by employing Eqs. (25.4) and (25.5). Thus, every specific problem belonging to a cluster becomes elementary after we have formulated and solved the appropriate generalized problem.

Consider, for instance, the following specific problem: *a flat loop in the form of an equilateral triangle with a side of length a is placed in a magnetic field whose induction varies according to the law* $\mathbf{B} = B_0 \sin \omega t \times \mathbf{i}$, *where* B_0 *and* ω *are constants, at right angles to* **B**; *find the induction emf in the loop at time t.*

This elementary problem can easily be solved via Eqs. (25.4) and (25.5):

$$\Phi = B_0 \sin wt \times A, \quad |\mathscr{E}_i| = B_0 A \omega \cos \omega t,$$

where $A = a^2\sqrt{3}/2$ is the area subtended by the loop.

We are now ready to formulate and solve problems belonging to the second cluster, which includes considering other physical phenomena related to phenomena discussed in problems of the first cluster. Let us consider, for example, the various phenomena related to the induction current in Example 25.1 (thermal, magnetic, etc.).

Suppose that *we are required to determine the amount of heat liberated in the loop during the first five seconds if the loop resistance R is* 0.5 Ω.

Ignoring the inductance and capacitance of the loop and allowing for Ohm's law, we find the induction current flowing in the loop:

$$I = \mathscr{E}_i/R = 2\beta a^2 t/R. \qquad (25.6)$$

Since this quantity varies in time, we must apply the DI method to find the sought amount of heat:

$$Q = \int_0^5 I^2 R \, dt = \int_0^5 \frac{4\beta^2 a^4 t^2}{R} \, dt = \frac{4\beta^2 a^4 t^3}{3R} \Big|_0^5. \qquad (25.7)$$

Substitution of numerical values yields $Q \approx 5.3 \times 10^{-5}$ J.

It is now easy to formulate and solve the following general problem of the second cluster: *a flat contour encompassing an area A and having an ohmic resistance R is placed in a magnetic field whose induction vector varies according to the law $\mathbf{B} = f(t)\,\mathbf{i}$, with $f(t)$ an arbitrary (differentiable) function of time t, in such a manner that the plane of the loop is at right angles to $\mathbf{B}$; find the amount of heat liberated by the loop during an arbitrary time interval t.*

Allowing for (25.4)-(25.7), we arrive at the solution to the generalized problem of the second cluster:

$$Q = \frac{A^2}{R} \int_0^t [f'(t)]^2 \, dt. \qquad (25.8)$$

Now any specific problem of the second cluster becomes elementary and can be solved by applying formula (25.8).

We are now ready to formulate and solve specific problems and the generalized problem of the third "cluster"

(and the fourth, fifth, and so on) by changing still further the conditions in which the physical phenomena of the first cluster proceed. Suppose that *in Example 25.1 the magnetic field is no more homogeneous.* Then Eq. (25.4) becomes invalid and we must employ the DI method to calculate the magnetic flux.

Example 25.2. *An infinitely long, straight conductor lies in the plane of a square loop with an ohmic resistance* $R = 7\Omega$ *and a side length* $a = 20$ cm *at a distance* $r_0 = 20$ cm *parallel to one of the loop's sides (Figure 25.1). The current flowing in the conductor varies according to the law* $I = \alpha t^3$, *with* $\alpha = 2$ A/s^3. *Find the current in the loop at time* $t = 10$ s.

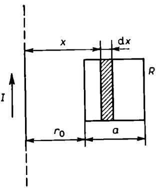

Figure 25.1

Solution. Owing to the change in the current flowing in the straight conductor, the magnetic flux that passes through the loop varies and an induction current is generated in the loop. The loop, therefore, finds itself in a nonhomogeneous magnetic field. Hence, we are forced to use the DI method to calculate the magnetic flux (see Example 23.9).

Let us partition the area subtended by the loop into strips so narrow that within each strip the magnetic field can be thought of as homogeneous (see Figure 25.1). The elementary magnetic flux passing through the narrow strip is

$$d\Phi = Ba\,dx = \frac{\mu_0 Ia\,dx}{2\pi x}.$$

Integrating this equation with respect to x from r_0 to $r_0 + a$, we find that

$$\Phi = \int_{r_0}^{r_0+a} \frac{\mu_0 Ia\,dx}{2\pi x} = \frac{\mu_0 a\alpha \ln(1 + a/r_0)}{2\pi} t^3.$$

Using Faraday's law (25.1), we can find the induction emf,

$$\mathscr{E}_i = \frac{3\mu_0 a\alpha \ln(1+a/r_0)}{2\pi} t^2,$$

and the current,

$$I = \frac{\mathscr{E}_i}{R} = \frac{3\mu_0 a\alpha \ln(1+a/r_0)}{2\pi R} t^2, \quad I \approx 2.4 \times 10^{-6} \text{ A}.$$

We could, of course, formulate and solve the generalized problem of the third cluster. But let us consider a specific problem belonging to the fourth.

Example 25.3. *The loop of Example 25.2 is moving away from the infinitely long, straight wire with a velocity* $v =$ 100 m/s *in the direction perpendicular to the conductor. A direct current* $I = 10$ A *is flowing in the conductor. Find the induction emf generated in the loop ten minutes after the loop started its motion if initially the loop was at a distance* $r_0 = 20$ cm *from the conductor.*

Solution. The current in the conductor remains constant, which means that the magnetic field generated by this current is time independent. However, the magnetic flux passing through the loop does not remain constant because the position of the loop in relation to the conductor changes. Let us find the flux as a function of time t. Applying the DI method, we get

$$\Phi = \frac{\mu_0 I a}{2\pi} \ln\left(1 + \frac{a}{x}\right), \tag{25.9}$$

with $x = vt + r_0$ the separation between conductor and loop at time t. If we now employ Faraday's law, in other words, find the time derivative of Φ, we have the induction emf generated in the loop:

$$\mathscr{E}_i = \frac{\mu_0 I a^2 v}{2\pi (a + vt + r_0)(vt + r_0)}.$$

Carrying out the necessary calculations (it is important to note that $vt \gg r_0$, and $vt \gg a$ for $t > 10^{-1}$ s, which means that we can ignore r_0 and a inside the parentheses), we obtain

$$\mathscr{E}_i = 8 \times 10^{-13} \text{ V}.$$

The numerical value of the induction emf is negligible because the loop's velocity is so high that, first, at time $t = 10$ s the loop is at a distance $x = 1$ km from the conductor and in this region the magnetic field is weak, and, second, the variation of the magnetic flux passing through the loop is low, too. Now let us change somewhat the terms of Example 25.3.

Example 25.4. *Suppose that in Example 25.3 only the side of the loop farthest from the conductor rather than the entire loop is moving away from the conductor with a velocity* **v** *(Figure 25.2). The length of the mobile side is a, the resistance of the loop is considered known, and the resistance of the leads and mobile side is assumed to be zero. Find the current generated in the loop at an arbitrary moment of time t.*

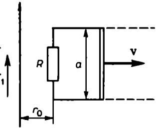

Figure 25.2

Solution. We denote the current in the infinitely long, straight conductor by I_1, which is constant (by hypothesis). The variation in the magnetic flux passing through the loop is caused by the motion of the moving side of the loop. Applying the DI method, we find the magnetic flux passing through the loop:

$$\Phi = \int_{r_0}^{vt} \frac{\mu_0 \mu I_1 a}{2\pi x}\, dx = \frac{\mu_0 \mu I_1 a}{2\pi} \ln\left(\frac{v}{r_0} t\right), \qquad (25.10)$$

which yields the following formulas for the induction emf and current:

$$\mathcal{E}_i = \frac{\mu_0 \mu I_1 a}{2\pi t}, \quad I = \frac{\mu_0 \mu I_1 a}{2\pi R t}.$$

We can now make the problem more complicated by assuming, for example, that the current in the conductor varies with time, $I_1 = f(t)$. Then, according to (25.10),

$$\Phi = \frac{\mu_0 \mu a f(t)}{2\pi} \ln\left(\frac{v}{r_0} t\right)$$

and, hence,

$$\mathscr{E}_i = \frac{\mu_0\mu a f'(t)}{2\pi} \ln\left(\frac{v}{r_0} t\right) + \frac{\mu_0\mu a f(t)}{2\pi t},$$

$$I = \frac{\mu_0\mu a f'(t)}{2\pi R} \ln\left(\frac{v}{r_0} t\right) + \frac{\mu_0\mu a f(t)}{2\pi R t}.$$

Example 25.5. *Along two smooth copper buses positioned at an angle α to the horizontal line there slides, owing to the force of gravity, a copper bar of mass m (Figure 25.3). The two upper ends of the buses are connected by a capacitor of capacitance C. The distance between the buses is l. The entire system is placed in a homogeneous magnetic field, with induction* **B** *at right angles to the plane in which the bar moves. The resistance of the buses, bar, and sliding contacts and the self-inductance of the loop are assumed negligible. Find the acceleration of the bar.*

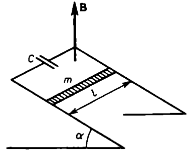

Figure 25.3

Solution. As in the previous example, the variation of the magnetic flux passing through the loop is caused by the movement of the bar. According to Ohm's law for a nonuniform section of a circuit, the induction emf $\mathscr{E}_i$ at any moment of time is equal to the potential difference $\Delta\varphi$ across the capacitor's plates:

$$\mathscr{E}_i = \Delta\varphi.$$

But $\Delta\varphi = Q/C$. Hence, the induction current in the loop is given by the following formula:

$$I = \frac{dQ}{dt} = C\frac{d(\Delta\varphi)}{dt} = C\frac{d\mathscr{E}_i}{dt}.$$

Since the magnetic field is homogeneous, we can write

$$\mathscr{E}_i = B\frac{dA}{dt} = Bl\frac{dx}{dt} = Blv,$$

where A is the area enclosed by the loop. Thus,

$$I = CBl\frac{dv}{dt} = CBla,$$

with a the sought acceleration of the bar.

Two forces act on the bar, the force of gravity mg and Ampere's force $IlB = CB^2l^2a$. By Newton's second law,

$$ma = mg \sin \alpha - CB^2l^2a.$$

Hence,

$$a = \frac{mg \sin \alpha}{m + CB^2l^2}.$$

If, in addition, there is friction that acts on the bar, we can easily show that

$$a = \frac{mg \sin \alpha - fmg \cos \alpha}{m + CB^2l^2},$$

where f is the coefficient of friction.

Let us now assume that there is no external magnetic field but the current I in the loop varies with time t. Then the magnetic flux generated by this current and passing through the loop,

$$\Phi = LI, \tag{25.11}$$

varies and there appears a self-induction emf

$$\mathscr{E}_{s-i} = -L\frac{dI}{dt}. \tag{25.12}$$

The self-induction emf generates a self-induction current. When an electric circuit is broken or closed there appears a break induced current

$$I = I_0 e^{-(R/L)t} \tag{25.13}$$

or a make induced current

$$I = I_0 (1 - e^{-(R/L)t}), \tag{25.14}$$

respectively, with $I_0 = \mathscr{E}_0/R$ the steady-state value of the current in the circuit and $\mathscr{E}_0$ the emf of the power source.

Example 25.6. *A solenoid with inductance $L = 10^{-1}$ H and resistance $R = 2 \times 10^{-2}\ \Omega$ is connected to a source of emf $\mathscr{E}_0 = 2$ V whose internal resistance is negligible.*

What amount of electricity will pass through the solenoid during the first five seconds after closure of the circuit?

Solution. When the circuit is closed, there appears a (variable) make induced current (25.14). To find the amount of electricity passing through the solenoid we employ the DI method.

Let us partition the time interval t into segments $\mathrm{d}t$ so small that within each the current may be considered to be approximately constant. Then the elementary amount of electricity $\mathrm{d}Q$ that passes through the solenoid during $\mathrm{d}t$ is given by the following formula:

$$\mathrm{d}Q = I\,\mathrm{d}t = \frac{\mathscr{E}_0}{R}\left(1 - e^{-(R/L)t}\right)\mathrm{d}t.$$

Integration with respect to t yields

$$Q = \int_0^5 \frac{\mathscr{E}_0}{R}\left(1 - e^{-(R/L)t}\right)\mathrm{d}t = \frac{\mathscr{E}_0}{R}\left(t + \frac{L}{R}e^{-(R/L)t}\right)\Big|_0^5,$$

$$Q \approx 181\ \mathrm{C}. \qquad (25.15)$$

If we had assumed, erroneously, that the current instantaneously reaches its steady-state value $I_0 = \mathscr{E}_0/R$ (which is, indeed, possible if L is small), we would have found that $Q = I_0 t = (\mathscr{E}_0/R)\,t$, $Q = 500$ C, which differs considerably from the correct answer given by (25.15). The answer $Q = 500$ C would be correct if the terms of the problem allowed us to neglect self-induction. Direct calculation shows that at $L = 10^{-3}$ H we would have $Q \approx 495$ C. Thus, at $L = 10^{-3}$ H self-induction can be ignored in the present problem.

26. Electromagnetic Oscillations

When studying electromagnetic oscillations, the common approach is to include in the physical system the electromagnetic field and the objects (which are of secondary importance) in which this field is localized (conductors, induction coils, capacitors, and the like).

The basic problem of the theory of electromagnetic oscillations is to find the law of variation in time

of an electric or magnetic physical quantity characterizing the process.

Using the equations that link this quantity with other quantities, we can find the values of these quantities, too.

Example 26.1. *Find the magnetic induction that exists within an ideal LC circuit at time* $t = 10^{-4}\ \pi/6$ s *if at* $t = 0$ *the charge on the capacitor was* $Q_1 = 10^{-5}$ C *and the current* I_1 *was zero. The inductance of the coil is* $L = 10^{-3}$ H, *the number of turns per unit of coil length is* $n = 10^3\ \mathrm{m}^{-1}$, *the capacitance of the capacitor is* $C = 10^{-5}$ F, *and the contour is placed in a vacuum.*

Solution. The physical system consists of the conductors forming the induction coil, the capacitor, and the electromagnetic field varying with time. We are seeking one of the parameters of this field (the magnetic induction) at a certain moment in time. This constitutes a basic problem of the theory of electromagnetic oscillations.

Let us find the law of variation of an electric or magnetic quantity. The physical process consists of free undamped (or natural) electromagnetic oscillations occurring in the circuit. As is known, the differential equation of such oscillations has the form

$$\ddot{Q} + \omega_0^2 Q = 0, \tag{26.1}$$

whose solution is given by the equation of harmonic oscillations

$$Q = Q_0 \sin(\omega_0 t + \alpha_0). \tag{26.2}$$

Note that equations similar to (26.1) and (26.2) can be written for other quantities (current, voltage, etc.). There are three unknown parameters in Eq. (26.2): the angular frequency ω_0, the amplitude Q_0, and the initial phase α_0. The angular frequency can be found from the equation

$$\omega_0^2 = 1/LC, \tag{26.3}$$

while the amplitude Q_0 and the initial phase α_0 can be found from the initial conditions ($Q = Q_1$ and $I_1 = -\,\mathrm{d}Q/\mathrm{d}t = 0$ at $t = 0$),

$$Q_1 = Q_0 \sin\alpha_0, \quad 0 = -\,Q_0\omega_0 \cos\alpha_0.$$

Whence, $\alpha_0 = \pi/2$ and $Q_0 = Q_1$. Thus, the equation of harmonic electromagnetic oscillations in the circuit has the form

$$Q = Q_1 \sin\left(\frac{1}{\sqrt{LC}}\, t + \frac{\pi}{2}\right). \tag{26.4}$$

We have, therefore, found the law of time variation of an electric or magnetic quantity (in our case, electric charge Q).

We can now calculate the value of the current flowing in the circuit at an arbitrary time t,

$$I = -\frac{dQ}{dt} = -\frac{Q_1}{\sqrt{LC}} \cos\left(\frac{1}{\sqrt{LC}}\, t + \frac{\pi}{2}\right), \quad I \approx 5 \times 10^{-2}\ \text{A},$$

and the magnetic induction,

$$B = \mu_0 \mu n I = -\frac{n\mu_0\mu Q_1}{\sqrt{LC}} \cos\left(\frac{1}{\sqrt{LC}}\, t + \frac{\pi}{2}\right),$$

$$B \approx 6.3 \times 10^{-5}\ \text{T}.$$

If we now use equations that link these quantities with other quantities, we can find any physical quantity characterizing the phenomenon. For instance, the potential difference between the plates of the capacitor is

$$\Delta\varphi = \frac{Q_1}{C} \sin\left(\frac{1}{\sqrt{LC}}\, t + \frac{\pi}{2}\right),$$

the electric field strength in the capacitor (assuming it is a plane-parallel capacitor with a plate area A) is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} = \frac{Q_1}{\varepsilon_0 A} \sin\left(\frac{1}{\sqrt{LC}}\, t + \frac{\pi}{2}\right),$$

the electric-field energy density inside the capacitor is

$$w = \frac{\varepsilon_0 \varepsilon E^2}{2} = \frac{\varepsilon Q_1^2}{2A^2\varepsilon_0} \sin^2\left(\frac{1}{\sqrt{LC}}\, t + \frac{\pi}{2}\right),$$

the magnetic-field energy density inside the coil is

$$w = \frac{B^2}{2\mu_0\mu} = \frac{\mu_0\mu Q_1^2 n^2}{2LC} \cos^2\left(\frac{1}{\sqrt{LC}}\, t + \frac{\pi}{2}\right),$$

etc. It is clear that if we abstract ourselves from the concrete numerical values in this example, we practically

have a ready solution to the generalized problem on free undamped electromagnetic oscillations in an LC circuit.

Example 26.2. *The ohmic resistance of an LC circuit is* $R = 10^2\ \Omega$, *the inductance* $L = 10^{-2}$ H, *and the capacitance* $C = 10^{-6}$ F. *Find the value of the current in the circuit at time* $t = 5\cdot 10^{-5}$ s *if at* $t = 0$ *the charge on the capacitor was* $Q_{01} = 10^{-5}$ C *and the current was zero.*

Solution. Electromagnetic oscillations occur in the *LC* circuit. To solve the basic problem of the theory of electromagnetic oscillations, we must find all the parameters (ω, δ, Q_0, and α_0) of the equation of damped oscillations:

$$Q = Q_0 e^{-\delta t} \sin(\omega t + \alpha_0). \tag{26.5}$$

The damping factor δ and the angular frequency ω can be found from the terms of the problem:

$$\delta = \frac{R}{2L}, \quad \omega = \sqrt{\frac{1}{LC} - \delta^2}.$$

This yields $\delta = 5 \times 10^3$ rad/s and $\omega \approx 8.7 \times 10^3$ rad/s.

The initial phase α_0 and amplitude Q_0 can be found from the initial conditions. Allowing for the fact that at $t = 0$ the charge on the capacitor Q is Q_{01}, we get the first equation for finding α_0 and Q_0:

$$Q_{01}t = Q_0 \sin \alpha_0. \tag{26.6}$$

The fact that at $t = 0$ the current

$$I = -\frac{dQ}{dt}$$

$$= -Q_0[-\delta e^{-\delta t} \sin(\omega t + \alpha_0) + \omega e^{-\delta t} \cos(\omega t + \alpha_0)] \tag{26.7}$$

is zero provides the second equation:

$$-\delta \sin \alpha_0 + \omega \cos \alpha_0 = 0. \tag{26.8}$$

Solving the system of equations (26.6) and (26.8), we find α_0 and Q_0:

$$\alpha_0 = \tan^{-1}(\omega/\delta), \quad \alpha_0 \approx \pi/3; \quad Q_0 = 2Q_{01}/\sqrt{3}.$$

Thus, we have the law of variation of charge Q with time (see Eq. (26.5)) in complete form:

$$Q = \frac{2Q_{01}}{\sqrt{3}} e^{-(R/2L)t} \sin\left(\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\, t + \frac{\pi}{3}\right).$$

We are now ready to determine any physical quantity that characterizes this specific physical phenomenon (damped electromagnetic oscillations). The sought value of the current can be found by solving Eq. (26.7):

$$I = Q_0 [\delta \sin(\omega t + \alpha_0) - \omega \cos(\omega t + \alpha_0)] e^{-\delta t},$$
$$I \approx 4.6 \times 10^{-2} \text{ A}.$$

As with problems on free electromagnetic oscillations, when dealing with problems on steady forced oscillations we must first determine the law of variation of an electric

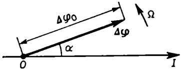

Figure 26.1

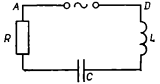

Figure 26.2

or magnetic quantity and then, using relationships that link different physical quantities, find the laws of variation of other sought quantities.

Often the phasor description is employed to solve problems on forced electromagnetic oscillations. In this approach a harmonic oscillation $\Delta\varphi = \Delta\varphi_0 \sin(\Omega t + \alpha)$ is represented by a vector $\Delta\boldsymbol{\varphi}$ called a phasor; the length of the vector is equal to $\Delta\varphi_0$ and the angle formed by the vector with a certain horizontal axis (the axis of currents I or the axis of voltages $\Delta\varphi$) is equal initially to the initial phase α (Figure 26.1). Phasor $\Delta\boldsymbol{\varphi}$ rotates counterclockwise with an angular velocity Ω. Let us examine several examples illustrating the use of the phasor method.

Example 26.3. *An electric circuit consists of a source of emf varying harmonically, an ohmic resistance R, a capacitance C, and an inductance L, all connected in series (Figure 26.2). Find the law of voltage variation on section $ARCLD$ as a function of time t.*

Solution. Let us employ the phasor method (Figure 26.3). Suppose that the law of variation of current is given in

the form

$$I = I_0 \sin \Omega t, \tag{26.9}$$

where Ω is the angular frequency (or rate) of variation of the external emf. We direct the current axis horizontally. Then the voltage variation on resistance R is depicted by a vector $\Delta\varphi_{0R}$ directed along the current axis, the voltage variation on inductance L by a vector $\Delta\varphi_{0L}$ directed at right angles to the current axis, and the voltage variation on capacitance C by a vector $\Delta\varphi_{0C}$ also directed at right angles to the current axis but in the direction opposite to that of $\Delta\varphi_{0L}$. The length of each vector is, respectively,

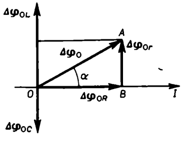

Figure 26.3

$$\Delta\varphi_{0R} = I_0 R, \quad \Delta\varphi_{0L} = I_0 \Omega L, \quad \Delta\varphi_{0C} = I_0/\Omega C.$$

The resultant voltage is depicted by the vector $\Delta\varphi_0 = \Delta\varphi_{0R} + \Delta\varphi_{0L} + \Delta\varphi_{0C}$. The sum of voltages on inductance and capacitance,

$$\Delta\varphi_{0r} \quad I_0 \left(\Omega L - \frac{1}{\Omega C}\right),$$

is known as the *reactive component of the voltage*. Thus, the net voltage varies according to the law

$$\Delta\varphi = \Delta\varphi_0 \sin(\Omega t + \alpha), \tag{26.10}$$

where the amplitude

$$\Delta\varphi_0 = I_0 \sqrt{R^2 + \left(\Omega L - \frac{1}{\Omega C}\right)^2} \tag{26.11}$$

and the initial phase

$$\alpha = \tan^{-1}\left(\frac{\Omega L - 1/\Omega C}{R}\right) \tag{26.12}$$

are found from the vector triangle OAB (see Figure 26.3).

Let us analyze Eq. (26.11). Only the amplitudes of the voltage and current, $\Delta\varphi_0$ and I_0 enter into this equation and not the instantaneous values $\Delta\varphi$ and I. Equa-

tion (26.11) shows that I_0 depends on the frequency Ω of the external emf. As Ω grows from zero to the value

$$\Omega_{res} = \omega_0 = 1/\sqrt{LC}, \qquad (26.13)$$

the amplitude of current, I_0, increases, since the *impedance*

$$Z = \sqrt{R^2 + \left(\Omega L - \frac{1}{\Omega C}\right)^2} \qquad (26.14)$$

of the circuit drops. At a value $\Omega = \Omega_{res}$ the amplitude of the current reaches its maximum value, the reactive component of the voltage vanishes, and the circuit behaves like a purely resistive circuit. This phenomenon is known as the *resonance of voltages*. From Eq. (26.12) it follows that in the event of a resonance of voltages the phase difference α between current oscillations and voltage oscillations vanishes. If Ω is further increased ($\Omega > \omega_0$), the amplitude of the current, I_0, decreases, asymptotically tending to zero.

Example 26.4. *A resistance $R = 10\ \Omega$ and an inductance $L = 0.1$ H are connected in series. What capacitance should be inserted in series into the circuit so that the phase shift between the emf and the current decreases by $\Delta\alpha = 27°$? The driving frequency of the external emf is $\nu = 50$ Hz.*

Solution. Let us employ the phasor approach. Figure 26.4 shows that

$$\tan \alpha_1 = \frac{I_0 \Omega L}{I_0 R} = \frac{\Omega L}{R}.$$

This yields $\alpha_1 = \tan^{-1}(\Omega L/R)$, or $\alpha_1 \approx 72°$. Hence $\alpha_2 = \alpha_1 - \Delta\alpha$, or $\alpha_2 = 45°$. By formula (26.12),

$$\tan \alpha_2 = \frac{\Omega L - 1/\Omega C}{R}.$$

This gives us the value of the sought capacitance (bearing in mind that $\Omega = 2\pi\nu$):

$$C = \frac{1}{2\pi\nu\,(2\pi\nu L - R)}, \quad C \approx 1.5 \times 10^2\ \mu\text{F}.$$

Example 26.5. *A section of a circuit consists of a capacitance $C = 200\ \mu$F and a resistance $R = 10^2\ \Omega$ connected in parallel. Find the impedance of the section if the driving frequency of the harmonic emf is $\nu = 50$ Hz.*

Solution. In the phasor method as applied to calculations of parallel circuits, the horizontal axis is the voltage axis (Figure 26.5). Then the current in the ohmic resistance, I_{0R}, coincides in phase with the voltage, and the

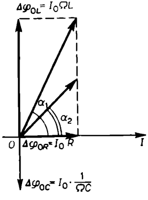

Figure 26.4

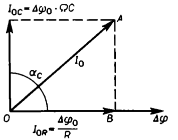

Figure 26.5

current flowing through the capacitance leads the voltage in phase by an angle $\alpha_C = 90°$. The net-current amplitude I_0 can be found from triangle OAB:

$$I_0 = \sqrt{(\Delta\varphi_0 \Omega C)^2 + \left(\frac{\Delta\varphi_0}{R}\right)^2} = \frac{\Delta\varphi_0}{R\,[1+(2\pi\nu CR)^2]^{-1/2}}.$$

This yields the impedance of the section:

$$Z = \frac{R}{\sqrt{1+(2\pi\nu CR)^2}}, \quad Z \approx 15.6\ \Omega.$$

Chapter 9
ELECTROMAGNETIC WAVES

27. Interference of Light

The basic problem in studying the interference of light is to calculate the interference pattern.

This means finding the distribution of intensity I of electromagnetic waves in space. Since intensity is proportional to the square of the amplitude E_0 of the electric field strength in the electromagnetic wave, the basic problem of the theory of interference is reduced to finding the amplitude E_0 of the resultant oscillation at an arbitrary point in the medium.

Most often, when calculating an interference pattern, it is necessary to determine the position of an arbitrary kth order maximum (or minimum) and the separation between two adjacent maxima (or minima). The method of solving the majority of problems on light interference can be reduced to two basic stages: finding the *optical path difference* δ and employing the *maximum condition*

$$\delta = k\lambda_0 \tag{27.1}$$

or the *minimum condition*

$$\delta = (2k+1)\frac{\lambda_0}{2}. \tag{27.2}$$

Example 27.1. *Calculate the interference pattern produced by two coherent sources* I *and* II (*Figure 27.1*) *positioned* $d = 5$ mm *apart at a distance* $L = 6$ m *from the screen. The wavelength of the light generated by the sources in a vacuum is* $\lambda_0 = 5 \times 10^{-7}$ m. *Also find the position of the fifth maximum on the screen and the distance between the adjacent maxima. The medium is a vacuum.*

Solution. Before meeting at an arbitrary point F on the screen (see Figure 27.1), at which point the result of interference is evaluated, each of the waves travels its own geometrical path, x_1 and x_2. For the sake of simplicity we will assume that the initial phases are equal to zero and the amplitudes are equal to each other. Then we can

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write the equations for the waves generated by the sources as follows:

$$E_1 = E_{01}\sin\left(2\pi\nu t - \frac{2\pi x_1}{\lambda_0}\right),$$

$$E_2 = E_{01}\sin\left(2\pi\nu t - \frac{2\pi x_2}{\lambda_0}\right).$$

According to the superposition principle, the resultant

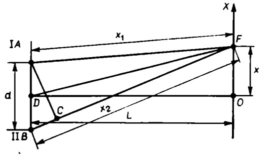

Figure 27.1

oscillation at point F,

$$E = E_1 + E_2$$

$$= 2E_{01}\cos\left[\frac{\pi}{\lambda_0}(x_1 - x_2)\right]\sin\left[2\pi\nu t - \frac{\pi}{\lambda_0}(x_1 - x_2)\right],$$

is harmonic oscillation with the same frequency ν but with an amplitude

$$E_0 = 2E_{01}\cos\left[\frac{\pi}{\lambda_0}(x_1 - x_2)\right] \tag{27.3}$$

that depends on a parameter, $(\pi/\lambda_0)(x_1 - x_2) = (\pi/\lambda_0)\,\delta$. Squaring (27.3), we obtain the distribution of the light intensity on the screen:

$$I = 4I_{01}\cos^2\left(\frac{\pi\delta}{\lambda_0}\right) = 2I_{01}\left[1 + \cos\left(\frac{2\pi\delta}{\lambda_0}\right)\right]. \tag{27.4}$$

Let us link the path difference δ with the coordinate x of point F on the screen. From the similarity of triangles ABC and DFO (assuming that $\delta \approx |BC|$ and $|FO| =$

x), we find that

$$\delta/d = x/L. \tag{27.5}$$

Hence.

$$\delta = (d\ L)\ x. \tag{27.6}$$

Thus, the intensity distribution is given by the following formula:

$$I = 2I_{01}\left[1+\cos\left(\frac{2\pi d}{\lambda_0 L}\,x\right)\right]. \tag{27.7}$$

The graph representing the function (27.7) is depicted in Figure 27.2. Allowing for the maximum conditions (27.1)

Figure 27.2

and (27.5), we can find the position of the kth maximum,

$$x_k = L\delta\ d = kL\lambda_0/d, \quad x_k = 3\times 10^{-3}\ \text{m}, \tag{27.8}$$

and the distance between adjacent maxima:

$$\Delta x = x_{k+1} - x_k = L\lambda_0/d, \quad \Delta x = 6\times 10^{-4}\ \text{m}. \tag{27.9}$$

Two real sources of light are not coherent. Hence, the above problem on the calculation of interference pattern produced by two coherent sources is ideal. However, the results and the method of solution are often employed in calculations of real interference devices. In the majority of such devices the beam of light is split into two coherent parts. After the parts of the initial beam have traveled different optical paths, they form an interference pattern.

Example 27.2. *A point-like source S of light with a wavelength* $\lambda_0 = 5\times 10^{-7}$ m *is placed at a distance* $r = 10$ cm *from the line of intersection of two flat mirrors, with the angle* α *between the mirrors equal to* $20'$ *(Fresnel mirrors). Determine the number of bright lines of the interference pat-*

tern formed on a screen that is l = 190 cm distant from the line of intersection of the mirrors (Figure 27.3).

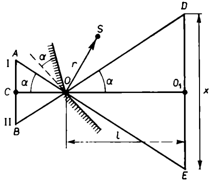

Figure 27.3

Solution. The interference pattern is formed by the light from two coherent sources I and II that are positioned at points A and B and are the virtual images of the source S of light in the two flat mirrors. This ideal problem has been solved in Example 27.1. Thus, to calculate the interference pattern we must determine the distance $|AB| = d$ between the sources. The distance from the sources to the screen is $L \approx l + r$. In triangle AOC the angle AOC is equal to α. Hence, $d = 2\,|AC| = 2\,|AO| \times \sin\alpha = 2r\alpha$, since $\sin\alpha \approx \alpha$ for small α's. Using formula (27.9), we can find the distance between adjacent bright lines:

$$\Delta x = \frac{\lambda_0 L}{d} = \frac{\lambda_0 (l+r)}{2r\alpha}.$$

The number of bright lines can be found if we determine the width of the interference pattern, and this is determined by the region where the waves generated by sources I and II overlap. Figure 27.3 shows that the width of the interference pattern is depicted by the interval $|DE| = x = 2\,|O_1D| = 2l\tan\alpha \approx 2\,l\alpha$. Dividing the width x of the interference pattern by the width Δx of a bright

line, we find the number N of bright lines:

$$N=\frac{x}{\Delta x}=\frac{4a^2rl}{\lambda_0(l+r)}, \qquad N\approx 26.$$

Example 27.3. *What must be the admissible width d_0 of the slits in Young's experiment so that a distinct interference pattern is formed on a screen S positioned at a distance L = 2 m from the slits (Figure 27.4)? The distance between the slits is d = 5 mm, and the wavelength of the light is $\lambda_0 = 5\times 10^{-7}$ m.*

Solution. In Young's experiment two slits—points A and B in Figure 27.4—are the coherent sources of light that form the interference pattern on screen S. Let us assume that these sources are point-like. Then the interference pattern can be calculated via formulas (27.8) and (27.9). If we shift the sources upward by a distance d_0, the interference pattern also is shifted by d_0. Let us now consider the interference pattern that is formed by the light of four point sources positioned at points A, A', B, and B'. This will consist of two interference patterns shifted in relation to each other by d_0. If this distance is smaller than the distance between the adjacent dark and bright lines, which according to (27.9) is equal to $\lambda_0 L/2d$, the total interference pattern is distinct.

Now suppose that we have two nonpoint-like coherent sources (slits of width $AA' = BB' = d_0$). According to what has just been said, the total interference pattern is distinct if

$$d_0\leqslant\frac{\lambda_0 L}{2d}, \text{ that is, } d_0\leqslant 0.1 \text{ mm}.$$

Example 27.4. *In the device for obtaining Newton's rings the space between the lens (index of refraction n_1 = 1.55) and the transparent flat plate (index of refraction n_3 = 1.50) is filled with a liquid with an index of refraction n_2 = 1.60 (Figure 27.5). The device is illuminated with monochromatic light ($\lambda_0 = 6\times 10^{-7}$ m) incident at right angles on the flat surface of the lens. Find the radius R of curvature of the lens if the radius of the fourth (k = 4) bright ring in transmitted light is ρ_k = 1 mm.*

Solution. Interference occurs in the thin liquid wedge (the index of refraction of the liquid, n_2, is greater than n_1 and n_3). In this thin liquid film of nonuniform thickness each ray of light splits into coherent rays. In transmitted light the kth maximum emerges because of the interference of ray I, which enters the plate at point A, and

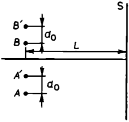

Figure 27.4

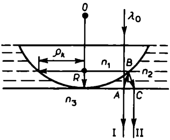

Figure 27.5

part II of the same ray ABC, which is reflected at points A and B and entering the plate at point C (see Figure 27.5). Since $n_2 > n_3$ and $n_2 > n_1$, there is no loss of a half-wave in the course of reflection at points A and B. Hence, the optical path difference acquired by rays I and II is

$$\delta = 2dn_2,$$

where d is the thickness of the liquid film at point A. Allowing for the fact that

$$d = \rho_k^2/2R$$

and the maximum condition (27.1), we find that

$$2\,\frac{\rho_k^2 n_2}{2R} = k\lambda_0,$$

which yields the following formula for the radius of curvature of the lens:

$$R = \frac{\rho_k^2 n_2}{k\lambda_0} \quad R \approx 66 \text{ cm}.$$

Example 27.5. *Light of wavelength $\lambda = 6 \times 10^{-7}$ m and a degree of monochromaticity $\Delta\lambda = 5 \times 10^{-10}$ m is incident on a plane-parallel glass plate whose index of refraction is $n = 1.5$. The angle of incidence is $i = 45°$. What is the maximum thickness of the plate at which the interference pattern in reflected light is still distinct?*

Solution. As is known, in the event of interference of monochromatic light (λ_0) in a thin film of thickness h and index of refraction n, the maximum condition has the form (in reflected light)

$$2h\sqrt{n^2 - \sin^2 i} = \left(k + \frac{1}{2}\right)\lambda. \qquad (27.10)$$

If the light is nonmonochromatic, the angular width of the kth interference maximum Δi is found from Eq. (27.10) by differentiating the left- and right-hand sides of this equation at $k = \text{const}$:

$$\frac{d}{di}\left(2h\sqrt{n^2 - \sin^2 i}\right)\Delta i = \left(k + \frac{1}{2}\right)\Delta\lambda,$$

whence

$$\Delta i = \frac{\left(k + \frac{1}{2}\right)\Delta\lambda}{\frac{d}{di}\left(2h\sqrt{n^2 - \sin^2 i}\right)}.$$

The angular separation δi of adjacent maxima for monochromatic light can also be found from Eq. (27.10) by differentiating the left- and right-hand side of this equation at $\lambda = \text{const}$:

$$\frac{d}{di}\left(2h\sqrt{n^2 - \sin^2 i}\right)\delta i = \lambda\delta k,$$

whence, at $\delta k = 1$ (adjacent maxima),

$$\delta i = \frac{\lambda}{\frac{d}{di}\left(2h\sqrt{n^2 - \sin^2 i}\right)}.$$

An interference pattern is distinct if $|\Delta i| < |\delta i|$, or

$$\left(k + \frac{1}{2}\right) < \frac{\lambda}{\Delta\lambda}. \qquad (27.11)$$

Substituting the expression for $k + 1/2$ that follows from Eq. (27.10) into (27.11), we find the maximum thickness of the film, h_{max}, at which an interference pattern can still be observed:

$$h_{max} \leqslant \frac{\lambda^2}{2\Delta\lambda\sqrt{n^2-\sin^2 i}}, \quad h_{max} \approx 0.27 \text{ mm}.$$

The degree of monochromaticity of laser light may be as high as $\Delta\lambda = 4 \times 10^{-12}$ m. Hence, to observe interference patterns in laser light we can take a plate of enormous thickness: $h_{max} \approx 3.3$ cm. The degree of monochromaticity of white (visible) light is $\Delta\lambda \approx 3.6 \times 10^{-7}$ m and, hence, in this case $h_{max} \approx 3.7 \times 10^{-7}$ m, that is, to observe interference patterns in white light we must take an extremely thin film, with a thickness of about a few tenths of a micrometer. A film of such thickness can be obtained in liquid or solid form.

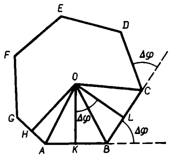

Figure 27.6

How can we calculate the interference pattern produced by light from many coherent sources of light rather than two? A common approach is to employ the *vector diagram method*. Let us take the simple case of equal amplitudes. We also assume that the phase difference of any two adjacent sources differs by the same value, $\Delta\varphi =$ const.

Figure 27.6 depicts a vector diagram corresponding to the addition of $N = 6$ oscillations with the same amplitudes,

$$|AB| = |BC| = |CD| = |DE| = |EF| = |FG| = E_{01}.$$

The amplitude of the resultant oscillation is depicted by segment $AG = E_0$. Let us find this quantity. Obviously, points A, B, C, D, E, F, and G lie on the circumference of a circle of radius $R = |OA| = |OB| = \ldots$. Let us

drop perpendiculars OK and OL from the center of the circle, O, onto segments AB and BC. Then $\measuredangle KOL = \Delta\varphi$ and, hence, $\measuredangle KOB = \Delta\varphi/2$. From triangle KOB, the radius of the circle is given by the following formula:

$$R = \frac{|KB|}{\sin(\Delta\varphi/2)} = \frac{E_{01}}{2\sin(\Delta\varphi/2)}. \tag{27.12}$$

Since $|AH| = |HG|$ ($OH \perp AG$ by construction), the resultant amplitude is

$$E_0 = |AG| = 2|AH| \tag{27.13}$$

Angle AOH is equal to $(1/2)(2\pi - N\Delta\varphi) = \pi - (1/2)N\Delta\varphi$, and from triangle AOH we find that

$$|AH| = R\sin\left(\pi - \frac{N\Delta\varphi}{2}\right) = R\sin\left(\frac{N\Delta\varphi}{2}\right).$$

Substituting this value of $|AH|$ into Eq. (27.13) and allowing for (27.12), we get

$$E_0 = E_{01}\frac{\sin(N\Delta\varphi/2)}{\sin(\Delta\varphi/2)}.$$

The energy of the oscillations (and the intensity I) is proportional to the square of the amplitude. Consequently, the intensity of the resultant oscillations is

$$I = I_{01}\frac{\sin^2(N\Delta\varphi/2)}{\sin^2(\Delta\varphi/2)}, \tag{27.14}$$

where I_{01} is the intensity of the light from one source.

For a small phase difference ($\Delta\varphi \to 0$), Eq. (27.14) assumes the form

$$I = I_{01}N^2.$$

Thus, *the intensity of the principal maximum in the interference pattern produced by the light from N sources is proportional to the square of the number of sources.*

28. Diffraction of Light

The basic problem in studying diffraction is to calculate the diffraction pattern, that is, find the distribution of light intensity I.

A more restricted problem is to find the position of the maxima and minima in the diffraction spectrum. Often the Fresnel zone method and the DI method (see Section 6) are employed in calculating diffraction patterns.

Example 28.1. *A plane monochromatic wave with a wavelength λ is incident at right angles on an infinitely long rectangular slit of width a (Figure 28.1). Find the distribution of the light intensity I in the diffraction pattern on*

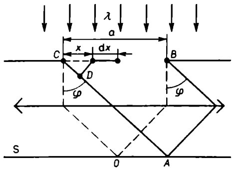

Figure 28.1

a screen S. Solve the same problem for a system of N parallel slits separated by opaque sections of width b (a diffraction grating).

Solution. Employing the Fresnel zone method, we can easily find the minimum condition for diffraction on a single slit,

$$a \sin \varphi = k\lambda, \tag{28.1}$$

and the maximum condition,

$$a \sin \varphi = (2k+1)\frac{\lambda}{2}. \tag{28.2}$$

This, however, does not provide us with the sought distribution of the light intensity I in the diffraction pattern. Let us employ the DI method. The zone of width dx (see Figure 28.1) that is x distant from the edge C of the slit sends a wave in the direction specified by an angle φ,

with the equation of this wave being

$$\mathrm{d}E = \mathrm{d}E_x \cos\left(\omega t - \frac{2\pi}{\lambda}\, x \sin\varphi\right), \qquad (28.3)$$

where

$$\mathrm{d}E_x = c'a, \quad c = \text{const}. \qquad (28.4)$$

In Eq. (28.3) we have allowed for the fact that for waves propagating in the direction CD the distances are reckoned on this straight line. Hence, $|CD| = x \sin\varphi$, and the phase of the wave emitted by zone $\mathrm{d}x$ is $\omega t - (2\pi/\lambda)\, x \sin\varphi$.

Integrating Eq. (28.4) over the entire slit for point O, we obtain the value of the arbitrary constant:

$$E_0 = \int_0^a \frac{c}{a}\,\mathrm{d}x = c.$$

Substituting the value of $\mathrm{d}E_x$ from (28.4) into Eq. (28.3), we find that

$$\mathrm{d}E = \frac{E_0}{a} \cos\left(\omega t - \frac{2\pi}{\lambda}\, x \sin\varphi\right) \mathrm{d}x. \qquad (28.5)$$

Integrating Eq. (28.5) over the entire slit, we get

$$E = \int_0^a \frac{E_0}{a} \cos\left(\omega t - \frac{2\pi}{\lambda}\, x \sin\varphi\right) \mathrm{d}x$$

$$= \left[E_0 \frac{\sin[(\pi/\lambda)\, a \sin\varphi]}{(\pi/\lambda)\, a \sin\varphi}\right] \cos\left(\omega t - \frac{\pi}{\lambda}\, a \sin\varphi\right).$$

Hence, the amplitude of oscillations at point A is

$$E_A = E_0 \frac{\sin[(\pi/\lambda)\, a \sin\varphi]}{(\pi/\lambda)\, a \sin\varphi}. \qquad (28.6)$$

Since light intensity is proportional to the square of the amplitude, Eq. (28.6) yields the expression for the distribution of light intensity on a screen in the event of diffraction on a single slit:

$$I_{\varphi_1} = I_0 \frac{\sin^2[(\pi/\lambda)\, a \sin\varphi]}{[(\pi/\lambda)\, a \sin\varphi]^2}. \qquad (28.7)$$

The reader can easily see that the minimum condition (28.1) follows from Eqs. (28.6) and (28.7).

A *diffraction grating* consists of N parallel slits, each of width a, separated by opaque segments, each of width b (Figure 28.2). The sum $a+b$ is known as the *grating space*.

To calculate a diffraction pattern obtained via a diffraction grating, let us employ the *Fresnel zone method*. We

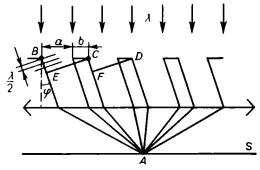

Figure 28.2

divide the wavefront of a plane monochromatic wave that is incident vertically on a diffraction grating (see Figure 28.2) in each slit into Fresnel zones by parallel planes, just as in the case of diffraction on a single slit. The distance between adjacent planes is $\lambda/2$. If into each slit there fits an even number of zones, then in the given direction (point A) a diffraction minimum forms. But if into each slit there fits an odd number of zones, each slit contains a nonextinguished zone. Suppose that these zones are positioned at the left edges of the slits (points B, C, D, etc.). The path difference between adjacent sources (nonextinguished zones) is constant:

$$\delta = |BE| = |CF| = \ldots = (a+b)\sin\varphi. \qquad (28.8)$$

To this path difference δ there corresponds a constant phase difference

$$\Delta\varphi = \frac{2\pi\delta}{\lambda} = \frac{2\pi(a+b)\sin\varphi}{\lambda}. \qquad (28.9)$$

Hence, the problem of calculating a diffraction pattern produced by a diffraction grating has been reduced to a problem of calculating the interference pattern produced by the light from many coherent sources with a constant phase difference (28.9). This has been solved in Section 27 by the vector diagram method. Allowing for (27.14) and (28.9), we obtain the distribution for the light intensity in a diffraction spectrum produced by a diffraction grating:

$$I_{\varphi N} = I_{\varphi 1} \frac{\sin^2\left[\frac{N\pi (a+b)\sin\varphi}{\lambda}\right]}{\sin^2\left[\frac{\pi(a+b)\sin\varphi}{\lambda}\right]}, \qquad (28.10)$$

where I_{φ_1} is the intensity generated by a single slit (see formula (28.7)).

Equation (28.10) provides the principal maximum condition:

$$(a+b)\sin\varphi = k\lambda. \qquad (28.11)$$

Example 28.2. *A plane monochromatic wave of wavelength* $\lambda_0 = 5 \times 10^{-7}$ m *is incident on a slit of width* $a = 10^{-2}$ mm *at right angles to the slit plane. Find the angular position of the first maximum in the diffraction pattern. The medium is a vacuum.*

Solution. The angular position of the first maximum can be found from the maximum condition (28.2). Hence,

$$\varphi = \sin^{-1}\left(\frac{3\lambda}{2a}\right), \quad \varphi \approx 4°18'. \qquad (28.12)$$

The angular position may be determined more precisely with the help of formula (28.7). Let us determine the extremum of the function $I_{\varphi_1} = I_{\varphi_1}(\varphi)$ by finding the first derivative with respect to φ and nullifying it:

$$0 = \frac{dI_{\varphi_1}}{d\varphi} = \frac{2I_0 \frac{\pi}{\lambda} a\cos\varphi \sin\left(\frac{\pi}{\lambda} a\sin\varphi\right)}{\left(\frac{\pi}{\lambda} a\sin\varphi\right)^3}$$

$$\times \left\{\cos\left(\frac{\pi}{\lambda} a\sin\varphi\right)\frac{\pi}{\lambda} a\sin\varphi - \sin\left(\frac{\pi}{\lambda} a\sin\varphi\right)\right\}.$$

This yields the following transcendental equation for finding the extremum values of φ:

$$\tan\left(\frac{\pi}{\lambda} a \sin\varphi\right) = \frac{\pi}{\lambda} a \sin\varphi,$$

which after introduction of the notation

$$m \equiv (a/\lambda) \sin\varphi \tag{28.13}$$

assumes the form

$$\tan \pi m = \pi m. \tag{28.14}$$

The roots of this transcendental equation are the following numbers:

$$m_1 = 1.43, \quad m_2 = 2.46, \quad m_3 = 3.47, \ \ldots \tag{28.15}$$

Combining (28.15) with Eq. (28.13), we can find the angular position of the first diffraction maximum:

$$\varphi = \sin^{-1}\left(\frac{1.43\lambda}{a}\right), \quad \varphi \approx 4°6'. \tag{28.16}$$

From (28.16) and (28.12) it follows that the more exact solution (28.16) differs considerably from the approximate solution (28.12). The error in the approximate solution can easily be calculated:

$$\varepsilon = \frac{\Delta\varphi}{\langle\varphi\rangle} 100\%, \quad \varepsilon = \frac{12' \times 100\%}{252'} \approx 5\%.$$

It should be noted that formula (28.7) allows us not only to find the exact angular position of the maxima in the diffraction pattern produced by a single slit but also to determine the intensity of these maxima.

Example 28.3. *Find the maximum order of a diffraction spectrum produced by a diffraction grating with a grating space* $a + b = 0.005$ mm *on which a plane monochromatic wave with wavelength* $\lambda_0 = 6 \times 10^{-7}$ m *is incident in a vacuum at right angles to the grating plane.*

Solution. The maximum order of the diffraction spectrum can be found from the maximum condition (28.11). Since the absolute value of $\sin\varphi$ cannot exceed unity, we have

$$a + b = k_{\max}\lambda_0.$$

Hence,

$$k_{max} = (a + b)/\lambda_0, \quad k_{max} \approx 8.$$

This equation is inexact, however. It was obtained on the assumption that in (28.10) the light intensity I_{φ_1} created by a single slit is constant and independent of angle φ. But Eq. (28.7) shows that I_{φ_1} depends on angle φ and can assume zero values for certain values of φ. The solution found determines only the maximum possible order of the spectrum. However, not all the principal maxima defined in (28.11) realize themselves: those whose position coincides with a minimum in the diffraction pattern from a single slit, (28.1), disappear; the only principal maxima that materialize are those that fall onto the central maximum of the diffraction pattern from a single slit. Hence, the maximum order of the realized principal maxima is determined by the combination of (28.1) with (28.11).

If in Eq. (28.1) we put $k = 1$, we find the angular half-width of the central maximum of the diffraction pattern from a single slit:

$$a \sin \varphi_{min} = \lambda. \qquad (28.17)$$

Using Eq. (28.11), we can find the maximum order of the realized principal maxima:

$$(a + b) \sin \varphi_{min} = k_{max}\lambda.$$

Allowing for condition (28.17), we get

$$k_{max} = (a + b)/a.$$

For the final solution of the problem we need to specify the width of a slit, a. For $a_1 = 10^{-3}$ mm and $a_2 = b = 2.5 \times 10^{-3}$ mm we obtain, via the last relationship, the following results:

$$k'_{max} = 5, \quad k''_{max} = 2.$$

Note that in the last solution we have neglected the realized principal maxima that fall onto the region of the maxima that follow the central maximum of the diffraction pattern from a single slit. It can be demonstrated that the intensity of these subsequent maxima is low.

Example 28.4. *The intensity of the central maximum in the diffraction pattern from a single slit is I_0. Find the ratios of the intensities of the three subsequent maxima to the intensity of the central maximum, I_0.*

Solution. From the maximum condition (28.13) for diffraction on a single slit,

$$(a/\lambda)\sin\varphi = m,$$

with $m = 1.43, 2.46, 3.47 \ldots$ (see (28.15)), and formula (28.7) we find the sought ratios:

$$\left(\frac{I_{\varphi_1}}{I_0}\right)' = \left[\frac{\sin(\pi m_1)}{\pi m_1}\right]^2, \left(\frac{I_{\varphi_1}}{I_0}\right)'' = \left[\frac{\sin(\pi m_2)}{\pi m_2}\right]^2, \left(\frac{I_{\varphi_1}}{I_0}\right)''' = \left[\frac{\sin(\pi m_3)}{\pi m_3}\right]^2.$$

Substitution of numerical values yields

$$(I_{\varphi_1}/I_0)' \approx 0.047, \quad (I_{\varphi_2}/I_0)'' \approx 0.017, \quad (I_{\varphi_1}/I_0)''' \approx 0.008.$$

THERMODYNAMICS AND KINETIC THEORY

Chapter 10
THERMODYNAMICS

29. The First Law of Thermodynamics

Experience shows that all macro-objects consist of micro-objects such as molecules, ions, atoms. Micro-objects are in a state of chaotic (thermal) motion. Since molecules, atoms and the like are extremely small, a medium-sized macro-object contains an enormous number of micro-objects. For example, in one cubic centimeter of ideal gas in standard conditions there are 2.7×10^{19} molecules. Hence, the physical systems that must be considered when solving problems of this section consist of a large number of objects. It can easily be shown that a dynamical (mechanistic) description of such systems is not only practically impossible but even meaningless. Therefore,

there are two methods of studying physical systems in molecular physics, methods mutually complementary, namely, the *thermodynamic* and the *statistical*. The statistical method is considered in Chapter 11.

The basis of the thermodynamic method is several fundamental laws derived from experience. First, the equation of state

$$f(p, V, T) = 0, \tag{29.1}$$

with p the pressure, V the volume, and T the thermodynamic temperature of the system. In this and the following chapter we consider only one physical system, the ideal, or perfect, gas. For an ideal gas the equation of state (29.1) is transformed into the ideal gas law

$$pV = \frac{m}{M} RT, \tag{29.2}$$

where m is the mass of the gas, M its molecular mass, and $R = 8.3144\ \mathrm{J \cdot mol^{-1}\ K^{-1}}$ the molar gas constant.

Equations of state (29.1) and (29.2) remain valid only for physical systems in a state of *thermodynamic equilibrium*. In such a state a physical system is characterized at each point of volume V by well-specified values of p and T that are the same for all points. Hence, a thermodynamically stable state of a physical system consisting of a large number of molecules is characterized by a small number of parameters (pressure p, volume V, temperature T, and a few other). These are known as *macro-parameters*, and the state of such a system is a *macro-state*. The concept of a thermodynamically stable (or equilibrium) state of a system is an idealized one. In any real case, the pressure p or temperature T at a point in volume V occupied by the system varies, but these variations (for an equilibrium state) must be so small that they can be ignored.

Another basic component of the thermodynamic method is the *first and second laws of thermodynamics*. By the first law of thermodynamics,

$$\delta Q = \mathrm{d}U + \delta A, \tag{29.3}$$

where

$$\delta Q = \frac{m}{M} C\, \mathrm{d}T \tag{29.4}$$

is the elementary amount of heat received by the system, C the molar heat capacity of the system, $\mathrm{d}U$ the variation of the internal energy of the physical system, and

$$\delta W = p\,\mathrm{d}V \tag{29.5}$$

is the elementary work done by the system. For an ideal gas,

$$\mathrm{d}U = \frac{m}{M}\,\frac{\iota R}{2}\,\mathrm{d}T, \tag{29.6}$$

where ι is the number of degrees of freedom of the molecules of the ideal gas.

The first law of thermodynamics in the form (29.3) is valid for elementary *quasi-static processes*. As a result of a quasi-static process the system passes through a sequence of equilibrium states. Since an equilibrium state can be depicted by a point in a certain system of coordinates (usually the p-V, or pressure-volume, coordinates), a quasi-static process is depicted in the same system of coordinates by a curve. Such diagramatic representation of various processes is very often used in solving problems by the thermodynamic method.

The following processes are assumed to be quasi-static: the *isochoric* (V = const, m = const), the *isobaric* (p = const, m = const), and the *isothermal* (T = const, m = const). Other processes, say, the *adiabatic* ($\delta Q = 0$), can also be quasi-static if they proceed so slowly that the system passes through a sequence of equilibrium states.

The amount of heat δQ and the work δW are characteristics of heat transfer and the capacity for work. These two processes are distinct: the first occurs on the micro-level as a result of the interaction of micro-objects (molecules, atoms, and the like), while the second occurs on the macro-level as a result of the interaction of macro-objects. Heat transfer is said to be elementary if the temperature variation $\mathrm{d}T$ is so small that the molar heat capacity C may be assumed constant. Then the amount of heat can be calculated by formula (29.4). A common way to calculate the amount of heat in the case of a nonelementary process of heat transfer is to employ the DI method

(see Section 6):

$$Q=\int_{T_1}^{T_2}\frac{m}{M}C\,\mathrm{d}T. \qquad (29.7)$$

To evaluate this integral we must know the dependence of C on other parameters.

If C is constant (C = const), the process is said to be *polytropic*, with

$$Q=\frac{m}{M}C\,(T_2-T_1) \qquad (29.8)$$

for such processes.

The process of performing work is said to be elementary if the variation of volume, $\mathrm{d}V$, is so small that the pressure p can be assumed to be constant. Of course, the pressure changes in an elementary process, but this variation $\mathrm{d}p$ must be so small that it can be ignored and the pressure can be considered approximately constant. Then work can be calculated by formula (29.5). For a nonelementary process work is calculated by employing the DI method:

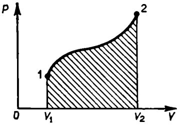

Figure 29.1

$$W=\int_{V_1}^{V_2}p\,\mathrm{d}V. \qquad (29.9)$$

In the p-V system of coordinates, work is numerically equal to the magnitude of the hatched area in Figure 29.1 (curve *1-2* depicts the corresponding process).

Thus, an elementary process for which the equation expressing the first law of thermodynamics in the form (29.3) must satisfy the two conditions formulated above.

For a nonelementary process the first law of thermodynamics is written in the form

$$\int_{T_1}^{T_2}\frac{m}{M}C\,\mathrm{d}T=\Delta U+\int_{V_1}^{V_2}p\,\mathrm{d}V, \qquad (29.10)$$

or, if we allow for (29.7) and (29.9),

$$Q = \Delta U + W, \qquad (29.11)$$

where $\Delta U = U_2 - U_1$ is the variation of the internal energy in the process (this variation does not depend on the type of process but only on the initial and final states of the physical system).

The basic problem of the thermodynamics of equilibrium processes is to find all the macro-states of a physical system.

If the initial and final states of a system are known, we can find the variation of the system's internal energy. If, in addition, we know the intermediate states (i.e. the process), we can find the work performed by the system, calculate the amount of heat received or liberated, and so on.

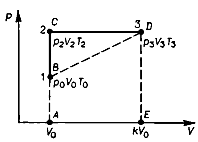

Figure 29.2

Example 29.1. *An amount of hydrogen contained in one cubic metre under standard conditions is first isochorically transferred to a state with a pressure that is n times higher than the initial pressure and then isobarically transferred to a state with a volume that is k times greater than the initial volume. Determine the variation of the internal energy of the gas, the work done by the gas, and the amount of heat received in the process.*

Solution. The physical system consists of a certain mass m (which can easily be calculated) of an ideal gas whose molecular mass M is known. The initial macro-state of the system (point *1* in Figure 29.2) is known (the standard pressure $p_0 \approx 10^5$ Pa, the standard temperature $T_0 = 273$ K, and the volume $V = 1$ m^3 specified in the terms of the problem). The states and processes in which the system takes part are depicted on a p-V diagram

(see Figure 29.2). Let us find the parameters of the second (point *2*) and third (point *3*) macro-states of the system. For this we use the ideal gas law (29.2) and the definitions of isoprocesses:

$$p_2 = np_0, \quad V_2 = V_0, \quad T_2 = \frac{Mp_2V_2}{mR} = \frac{Mnp_0V_0}{mR}, \quad (29.12)$$

$$p_3 = p_2 = np_0, \; V_3 = kV_0, \; T_3 = \frac{Mp_3V_3}{mR} = \frac{Mnkp_0V_0}{mR}. \quad (29.13)$$

Since the processes in which the system participates are quasi-static and polytropic, the sought quantities can be found by using the formulas found earlier. The variation in internal energy is

$$\Delta U = \frac{m}{M}\,\frac{iR}{2}(T_3 - T_0) = \frac{miR}{2M}\left(\frac{Mnkp_0V_0}{mR} - \frac{Mp_0V_0}{mR}\right)$$
$$= \frac{i}{2}\,p_0V_0\,(nk-1). \quad (29.14)$$

In an isochoric process $dV = 0$ and no work is done. The work done in the isobaric process is

$$W = p_2\,(V_3 - V_2) = np_0\,(kV_0 - V_0) = p_0V_0n\,(k-1). \quad (29.15)$$

The amount of heat

$$Q = Q_1 + Q_2 = \frac{m}{M}C_V\,(T_2 - T_0) + \frac{m}{M}C_p\,(T_3 - T_2)$$
$$= \frac{p_0V_0}{2}\,[i\,(nk-1) + 2n\,(k-1)], \quad (29.16)$$

where we have allowed for Mayer's formula

$$C_p = C_V + R. \quad (29.17)$$

The amount of heat can be calculated by using the formula (29.11), which expresses the first law of thermodynamics:

$$Q = \Delta U + W = \frac{i}{2}\,p_0V_0\,(nk-1) + p_0V_0n\,(k-1)$$
$$= \frac{p_0V_0}{2}\,[i\,(nk-1) + 2n\,(k-1)];$$

this coincides with the earlier result (29.16). To carry out a numerical calculation we must select reasonable values for n and k. Equations (29.13) show that the value of n

determines the maximum value of pressure, $p_3 = np_0$, while the product nk determines the maximum temperature, $T_3 = nkT_0$. The value of n cannot exceed $n_{max} = 100$, since at pressures equal to (or greater than) 100 p_0 the gas ceases to be ideal. The product nk cannot exceed the value $(nk)_{max} \approx 10$ since at temperature $T_3 \approx 10T_0 \approx 3 \times 10^3$ K (and higher) the walls of the vessel containing the hydrogen could melt (they must be cooled) and molecular hydrogen transforms into atomic hydrogen; at still higher temperatures the atomic hydrogen transforms into hydrogen plasma. If we put $n = 5$ and $k = 2$, we find that Eqs. (29.14), (29.15), and (29.16) yield

$$\Delta U \approx 2.2 \times 10^6 \text{ J}, \quad W = 5 \times 10^5 \text{ J}, \quad Q \approx 2.7 \times 10^6 \text{J}.$$

It is advisable to study the following problem: what must be the relationship between n and k (at nk = const) if we want the ratio W/Q to be maximal? But we will leave this problem for the reader to solve. Instead we pose the following problem: how do the sought quantities change if the system proceeds from the initial state to the final state quasi-statically along the dashed straight line in Figure 29.2? The answer can easily be found if we study formula (29.11) for the first law of thermodynamics. The variation of internal energy ΔU, does not depend on the type of process but on the initial and final states of the system. Hence, the variation in internal energy does not change: $\Delta U \approx 2.2 \times 10^6$ J. The amount of work done by the system will decrease (the area of the trapezoid $ABDE$ is smaller than the area of the rectangle $ACDE$). Hence, according to the first law of thermodynamics (29.11), the system will receive less heat.

Example 29.2. *Two moles of nitrogen, N_2, are under standard conditions. They are then transformed isothermically into a certain state and then quasi-statically and adiabatically into a finite state with a volume that is four times the initial one. Find the work performed by the gas if Q =* 11 300 J *of heat was transmitted to the gas in the isothermal process.*

Solution. Let us determine the intermediate and final macro-states of the system. The ideal gas law results in

an indeterminate system of equations. Let us employ the first law of thermodynamics in the form (29.11). For the isothermal process,

$$Q = W_1 = \int_{V_0}^{V_2} p\,\mathrm{d}V = \int_{V_0}^{V_2} \frac{m}{M} RT_0 \frac{\mathrm{d}V}{V} = \frac{m}{M} RT_0 \ln \frac{V_2}{V_0},$$

where W_1 is the work done by the gas in the isothermal process. This enables us to find the volume that the gas must occupy in the intermediate state:

$$V_2 = V_0 \mathrm{e}^{QM/mRT}.$$

Next, using the equation governing an adiabatic process,

$$T_0 V_2^{\gamma-1} = T_3 (4V_0)^{\gamma-1},$$

with $\gamma = C_p/C_V$ the *molar heat capacity ratio* (commonly known as the *specific heat ratio*), we can find the final temperature:

$$T_3 = T_0 \frac{\mathrm{e}^{QM(\gamma-1)/mRT}}{4^{\gamma-1}}.$$

Formula (29.6) yields the following expression for the internal-energy variation:

$$\Delta U = \frac{m}{M} \frac{iR}{2} (T_3 - T_0) = \frac{m}{M} \frac{iRT_0}{2} \left[\frac{\mathrm{e}^{QM(\gamma-1)/mRT}}{4^{\gamma-1}} - 1 \right].$$

This gives the following result: $W \approx 4500$ J. Note that after finding the parameters of the macro-states we could have calculated the sought amount of work directly by employing formula (29.5).

Example 29.3. *For an ideal gas find the equation of a process in which the heat capacity of the gas varies with temperature according to the law* $C = \alpha/T$, *with* $\alpha =$ const.

Solution. The process is not polytropic. Hence, we can apply the first law of thermodynamics in the form (29.3) for one mole of the gas:

$$\frac{\alpha}{T} \mathrm{d}T = C_V\,\mathrm{d}T + p\,\mathrm{d}V.$$

Using the ideal gas law (29.2), we can rewrite this equation:

$$\frac{\alpha}{T}\,\mathrm{d}T = C_V\,\mathrm{d}T + RT\,\frac{\mathrm{d}V}{V}.$$

Dividing the left- and right-hand sides by RT and integrating, we get

$$-\frac{\alpha}{RT} = \frac{1}{\gamma-1}\ln T + \ln V + \text{const}.$$

This gives us the sought equation of the process:

$$VT^{1/(\gamma-1)}e^{\alpha/RT} = \text{const}.$$

30. The Second Law of Thermodynamics

As a result of some process a system may return to its initial state. Such a process is known as *cyclic*. Using the first law of thermodynamics, we can prove that the thermal efficiency of an arbitrary cycle is

$$\eta = (Q_1 - Q_2)/Q_1, \tag{30.1}$$

where Q_1 is the amount of heat received by the system from the heater, and Q_2 the heat rejected by the system to the cooler. For the Carnot cycle (two isotherms and two adiabatic curves) we have

$$\eta = (T_1 - T_2)/T_1, \tag{30.2}$$

where T_1 is the temperature of the heater, and T_2 the temperature of the cooler.

The ratio $\delta Q\ T$ is known as the *reduced heat of an elementary process*. According to Clausius' theorem,

> the sum of reduced heats for an arbitrary cycle is a negative quantity, and for a reversible cycle it is zero:

$$\oint \frac{\delta Q}{T} \leqslant 0. \tag{30.3}$$

Consequently, it follows that

> the sum of reduced heats, $\int_1^2 T^{-1}\delta Q$, for any reversible process does not depend on the type of process but is

determined solely by the initial (*1*) and final (*2*) states of the system.

Next we introduce the notion of *entropy* S of a system as a state function whose variation depends only on the initial and final states of the system in the following manner:

$$S_2 - S_1 = \int_1^2 \frac{\delta Q}{T}, \qquad (30.4)$$

where integration is carried out over any reversible process as a result of which the system goes from state *1* to state *2*.

Example 30.1. *The cycle depicted in Figure 30.1 consists of two isotherms* ($T_1 = 600$ K *and* $T_2 = 300$ K) *and two isobars* ($p_1 = 4p_2$). *Determine the thermal efficiency of the cycle if the working substance is an ideal gas whose molecules have five degrees of freedom* ($i = 5$).

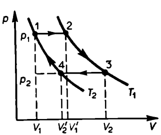

Figure 30.1

Solution. The physical system consists of one mole of an ideal gas. A cyclic process consisting of two isotherms and two isobars (see Figure 30.1) occurs in the system. To find the efficiency of the cycle using formula (30.1), we must determine Q_1 and Q_2. The system receives an amount of heat Q_1 in the isobaric transition from state *1* with parameters p_1, V_1, T_2 to state *2* with parameters p_1, V_1', T_1 and in the isothermal expansion from state *2* to state *3* with parameters p_2, V_2, T_1:

$$Q_1 = C_p (T_1 - T_2) + RT_1 \ln (V_2/V_1). \qquad (30.5)$$

The system rejects an amount of heat Q_2 in the isobaric transition from state *3* to state *4* with parameters p_2, V_2', T_2 and in the isothermal compression from state *4* to state *1*:

$$Q_2 = C_p (T_1 - T_2) + RT_2 \ln (V_2'/V_1). \qquad (30.6)$$

From Boyle's law for the isotherms T_1 and T_2,

$$p_1V_1' = p_2V_2 \quad \text{and} \quad p_1V_1 = p_2V_2',$$

it follows that

$$\frac{V_2}{V_1'} = \frac{V_2'}{V_1} = \frac{p_1}{p_2}.$$

Substituting these volume ratios into (30.5) and (30.6) and allowing for (30.1), we obtain

$$\eta = \frac{R(T_1 - T_2)\ln(p_1/p_2)}{C_p(T_1 - T_2) + RT_1\ln(p_1/p_2)}.$$

Using the well-known relationship $C_p = (1/2)(i + 2)R$, with i the number of degrees of freedom, we finally get

$$\eta = \frac{T_1 - T_2}{T_1 + \dfrac{(i+2)(T_1 - T_2)}{2\ln(p_1/p_2)}}.$$

Calculation yields $\eta \approx 22.5\%$.

The thermal efficiency of a Carnot cycle with the same temperatures T_1 of the heater and T_2 of the cooler is

$$\eta = (T_1 - T_2)/T_1, \quad \eta \approx 50\%.$$

If the degree of compression is increased (say by putting $p_1/p_2 = 20$) and the number of degrees of freedom of the gas molecules is decreased (say, to $i = 3$), the thermal efficiency of a cycle consisting of two isotherms and two isobars can be increased by up to $\eta \approx 35\%$. But in all cases it remains lower than the efficiency of the respective Carnot cycle.

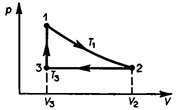

Figure 30.2

Example 30.2. *A cycle consists of an isotherm* ($T_1 =$ 600 K), *an isobar, and an isochor (Figure 30.2). The volume ratio* V_2/V_3 *is equal to two. The working substance is an ideal gas with* $i = 5$. *Determine the efficiency of the cycle as a function of the maximum* (T_1) *and minimum temperatures of the working substance.*

Solution. Let us find the minimum temperature. In the isobaric process the gas cools off and in the isochoric process it heats up. Hence, the minimum temperature is the one which the substance has in state *3*, or T_3. From the ideal gas laws for states *2* and *3*,

$$pV_2 = RT_1 \quad \text{and} \quad pV_3 = RT_3,$$

we can find the minimum temperature:

$$T_3 = T_1 V_3/V_2 = 0.5T_1, \quad T_3 = 300 \text{ K}.$$

Since all the processes in the cycle are polytropic, we can use the first law of thermodynamics for the isothermal process and formula (29.8) for the isobaric and isochoric processes to find the amount of heat absorbed by the working substance (or rejected by it) in these processes. For the isothermal process,

$$Q_{12} = RT_1 \ln (V_2/V_1).$$

Since $V_1 = V_3$ and $V_2/V_3 = T_1/T_3$, we have

$$Q_{12} = RT_1 \ln (T_1/T_3).$$

For the isobaric process,

$$Q_{23} = C_p (T_1 - T_3) = \frac{i+2}{2} R (T_1 - T_3).$$

Finally, for the isochoric process,

$$Q_{31} = C_V (T_1 - T_3) = \frac{iR}{2} (T_1 - T_3).$$

Combining these results according to formula (30.1), we find the efficiency of the cycle:

$$\eta = \frac{Q_{12}+Q_{31}-Q_{23}}{Q_{12}+Q_{31}} = 1 - \frac{(1/2)(i+2)(T_1-T_3)}{T_1 \ln (T_1/T_3) + (1/2)\, i\, (T_1-T_3)},$$

$$\eta \approx 10\%.$$

Example 30.3. *Find the variation of the entropy on one mole of an ideal gas in the isobaric, isochoric, and isothermal processes.*

Solution. The physical system, one mole of an ideal gas, participates in three isoprocesses. The processes are quasi-static and reversible. Hence, the entropy variation

can be found directly by employing formula (30.4). For the isobaric process,

$$\Delta S_p = \int_{T_1}^{T_2} \frac{\delta Q}{T} = \int_{T_1}^{T_2} \frac{C_p \, dT}{T} = C_p \ln \frac{T_2}{T_1} = C_p \ln \frac{V_2}{V_1}. \quad (30.7)$$

For the isochoric process,

$$\Delta S_V = \int_{T_1}^{T_2} \frac{\delta Q}{T} = \int_{T_1}^{T_2} \frac{C_V \, dT}{T} = C_V \ln \frac{T_2}{T_1} = C_V \ln \frac{p_2}{p_1}. \quad (30.8)$$

Finally, for the isothermal process,

$$\Delta S_T = \int \frac{\delta Q}{T} = \int \frac{\delta W}{T} = \int \frac{p \, dV}{T} = \int_{V_1}^{V_2} \frac{RT \, dV}{TV} = R \ln \frac{V_2}{V_1}. \quad (30.9)$$

Let us now apply the above results to the cycle of Example 30.2. The cycle consists of an isobar, isochore, and isotherm. All the processes are reversible, and so is the cycle. According to Clausius' law (30.3), the variation in entropy in a reversible cycle is zero:

$$\Delta S_p + \Delta S_V + \Delta S_T = 0.$$

Hence, allowing for (30.7)-(30.9) and the notation used in Example 30.2, we find that

$$C_p \ln \frac{T_3}{T_1} + C_V \ln \frac{T_1}{T_3} + R \ln \frac{V_2}{V_1} = 0.$$

Since $V_1 = V_3$ and $V_2/V_3 = T_1/T_3$ (see Example 30.2), we have

$$-C_p \ln \frac{T_1}{T_3} + C_V \ln \frac{T_1}{T_3} + R \frac{T_1}{T_3} = 0,$$

which leads to the well-known Mayer relation (29.17):

$$C_p = C_V + R.$$

We depict the entropy variation in the cycle using the S-ln T coordinates (Figure 30.3). On the segment *1-2* of

isothermal expansion ($\ln T = \text{const}$) the entropy increased by $\Delta S_T = 0.7R$; on the segment *2-3* of isobaric cooling the entropy decreased by $\Delta S_p = 3.5 \times 0.7R$; finally, on the segment *3-1* of isochoric heating the entropy increased by $\Delta S_V = 2.5 \times 0.7R$. The total entropy variation in the cycle is zero: $\Delta S_p + \Delta S_V + \Delta S_T = 0$. The increase in entropy on segments *3-1* and *1-2* seems natural (this agrees with the general law of entropy increase), but at first glance it seems that the entropy on segment *2-3* should increase too (according to the same

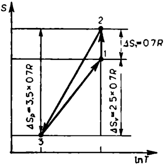

Figure 30.3

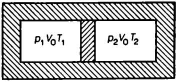

Figure 30.4

law). However, there is no contradiction here. The law is valid only for an adiabatically isolated system, and the system considered in Example 30.2 is not adiabatically isolated: during isothermal expansion *1-2* and isochoric heating *3-1* it receives heat, while during isobaric cooling *2-3* it rejects heat to external objects.

Example 30.4. *An adiabatically isolated vessel is divided into two equal parts by a rigid partition that does not conduct heat (Figure 30.4). Each half of the vessel is filled with one mole of the same ideal triatomic gas: at* $T_1 = 600$ K *in the left half and at* $T_2 = 300$ K *in the right. Then the partition is removed from the vessel. Determine the entropy variation in the gas after an equilibrium state sets in.*

Solution. Let us consider three physical systems. System I consists of one mole of the gas in the left half of

the vessel at a temperature T_1. Prior to removal of the partition the gas in the left half is adiabatically isolated and in an equilibrium state. System II consists of one mole of the same gas in the right half of the vessel at a temperature $T_2 < T_1$. Prior to removal of the partition the gas in the right half is also adiabatically isolated and in an equilibrium state. System III (the final one) appears because the partition between systems I and II has been removed. Prior to removal and after removal in systems I and II and in system III, respectively, there occur quasi-static and irreversible processes as a result of which equilibrium states set in in all three.

Since the processes are irreversible, we cannot use formula (30.4) to find the entropy variation in system III directly. What is needed are processes that will take systems I and II from the initial state to the final state reversibly. For this the initial adiabatic isolation of systems I and II must be violated. Suppose that instead of the partition that cannot conduct heat we take a partition that is massless and conducts heat ideally. Then systems I and II are in thermal contact, that is, they are no more adiabatically isolated. In each there occurs a (reversible) isochoric process: isochoric cooling in the left half of the vessel and isochoric heating in the right half to a common temperature θ. The final equilibrium temperature can easily be found:

$$\theta = (T_1 + T_2)/2.$$

The next step is to remove the modified partition. Since both subsystems, I and II, are in a state of thermodynamic equilibrium at a temperature θ, the overall system III is also in a state of thermodynamic equilibrium. Note that while the isochoric processes occurring in systems I and II can be considered reversible, the process of heat transfer in system III cannot be thought of as reversible. Let us denote the entropy variations in systems I and II by ΔS_1 and ΔS_2, respectively. Then the variation of entropy in system III is

$$\Delta S = \Delta S_1 + \Delta S_2.$$

According to (30.8), for system I we have

$$\Delta S_1 = \int_{T_1}^{\theta} \frac{C_V\, dT}{T} = C_V \ln \frac{\theta}{T_1} = C_V \ln \frac{1}{2}\left(1 + \frac{T_2}{T_1}\right),$$

and for system II we have

$$\Delta S_2 = \int_{T_2}^{\theta} \frac{C_V\, dT}{T} = C_V \ln \frac{\theta}{T_2} = C_V \ln \frac{1}{2}\left(1 + \frac{T_1}{T_2}\right).$$

It is easy to see that ΔS_1 is negative and ΔS_2 positive, that is, the entropy of system I decreases and that of system II increases (the reader will recall that these systems ceased to be adiabatically isolated after a heat-conducting partition was inserted instead of the initial one; hence, the entropy of each can increase or decrease). The overall system III remains adiabatically isolated and its entropy must increase as a result of an irreversible process. Indeed, $|\Delta S_1| < |\Delta S_2|$ and

$$\Delta S = \Delta S_1 + \Delta S_2$$
$$= C_V\left[\ln \frac{1}{2}\left(1 + \frac{T_2}{T_1}\right) + \ln \frac{1}{2}\left(1 + \frac{T_1}{T_2}\right)\right] > 0.$$

Since $C_V = iR/2$, we have $\Delta S \approx 3\ \mathrm{J \cdot mol^{-1} K^{-1}}$. Thus, if the amount of heat Q_1 rejected by system I is equal to the amount of heat Q_2 received by system II ($|Q_1| = |Q_2|$), the absolute value of the entropy variation ΔS_1 in system I is not equal to the absolute value of the entropy variation ΔS_2 in system II in the same heat transfer process ($|\Delta S_1| \neq |\Delta S_2|$).

Chapter 11
KINETIC THEORY

31. The Maxwell-Boltzmann Distribution

In the *statistical method*, in contrast to the thermodynamic, an essential assumption concerns the "granular" structure of macro-objects. This method employs the following propositions (corroborated by numerous experiments):

all macro-objects consist of micro-objects, which participate in chaotic motion and interact with each other.

In *classical statistical physics* it is assumed that

no two similar macro-objects are identical.

The behavior of a single micro-object (a particle) is studied in the six-dimensional phase space (μ-space) of three position coordinates (x, y, z) and three projections of momentum (p_x, p_y, p_z) or three projections of velocity (v_x, v_y, v_z). The state of a single micro-object is specified by a point in this space. If the micro-object moves chaotically, its occurrence inside a volume element $d\tau = dx\,dy\,dz\,dp_x\,dp_y\,dp_z$ in this space constitutes a random event whose probability is

$$dw = f(x, y, z, p_x, p_y, p_z)\,d\tau, \qquad (31.1)$$

with f the distribution function (*the probability density*). The function f satisfies the normalization condition

$$\int f\,d\tau = 1. \qquad (31.2)$$

In (31.2) integration is carried out over the entire phase space. Using the concept of a distribution f, we can define the mean value of a function $\varphi(x, y, z, p_x, p_y, p_z)$ thus:

$$\langle\varphi\rangle = \int f\varphi\,d\tau. \qquad (31.3)$$

The *Maxwell-Boltzmann distribution* of molecules in the μ-space has the form

$$dw(x, y, z, p_x, p_y, p_z)$$
$$= e^{-\left[\frac{p_x^2+p_y^2+p_z^2}{2m}+U(x, y, z)\right]\Big/kT}\,dx\,dy\,dz\,dp_x\,dp_y\,dp_z, \qquad (31.4)$$

where $U(x, y, z)$ is the potential energy of a molecule, and m the molecule's mass.

The Maxwell-Boltzmann distribution can be thought of as two independent distributions in a three-dimensional momentum space (*the Maxwell distribution*),

$$dw(p_x, p_y, p_z) = Ae^{-(p_x^2+p_y^2+p_z^2)/2mkT}\,dp_x\,dp_y\,dp_z, \qquad (31.5)$$

and in a three-dimensional coordinate space (*the Boltzmann distribution*),

$$dw(x,\ y,\ z) = Be^{-U(x,\ y,\ z)/2kT}\,dx\,dy\,dz, \qquad (31.6)$$

where A and B are constants that can be found from the normalization condition (31.2). Allowing for the normalization condition (31.2), we arrive at the Maxwell distribution in the following form:

$$dw(p_x,\ p_y,\ p_z) = (2\pi mkT)^{3/2}\,e^{-(p_x^2+p_y^2+p_z^2)/2mkT}\,dp_x\,dp_y\,dp_z, \qquad (31.7)$$

which makes it possible to find the distribution in velocity components $(v_x,\ v_y,\ v_z)$,

$$dw(v_x,\ v_y,\ v_z) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}}\,dv_x\,dv_y\,dv_z, \qquad (31.8)$$

the distribution in speed (v),

$$dw(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}\,dv, \qquad (31.9)$$

the distribution in kinetic energy (E_k),

$$dw(E_k) = 2\pi\left(\frac{1}{\pi kT}\right)^{3/2} E_k^{1/2}\,e^{-\frac{E_k}{kT}}\,dE_k, \qquad (31.10)$$

and other distributions.

When a system consisting of N particles is in thermodynamic equilibrium, its macro-state is characterized by a relatively small number of macro-parameters (physical quantities that can be found from measurements in experiments) having definite, time-independent values. Owing to the chaotic motion of the particles their position and velocity change constantly. This means that while the macro-parameters remain unchanged, the micro-parameters vary. Thus, a single macro-state has corresponding to it a multitude of micro-states, which implies that any macroscopic quantity depends on the microscopic parameters. In statistical physics it is assumed that

> physical quantities observable in experiments (macro-parameters) can be found as mean values calculated over the set of admissible micro-states

(see (31.3)). Consequently, one of the main problems solved by the statistical method is finding mean values of various physical quantities and determining the mean number of particles dN (belonging to a collection of N particles) that possess a certain property.

Example 31.1. *Nitrogen is in a vessel at a pressure $p =$ 1 atm and a temperature $T = 300$ K. Find the fractional number of nitrogen molecules whose speed lies within the interval ranging from $\langle v\rangle$ to $\langle v\rangle + dv$, with $dv = 1$ m/s. No external forces are present.*

Solution. At 1 atm and 300 K nitrogen may be assumed to be an ideal gas. In the absence of external forces,

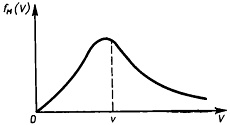

Figure 31.1

the molecules of an ideal gas obey the Maxwell distribution. The concrete form of this distribution is determined by the terms of the problem, in short, we must use the Maxwell distribution in the absolute value of the velocity, (31.9):

$$dN = N4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}\, dv, \qquad (31.11)$$

where dN is the number of molecules (of the given N molecules) whose speed lies within the interval from v to $v + dv$, and m is the mass of a nitrogen molecule. As is known, formula (31.11) is valid if dv is so small that any variation of the distribution function

$$f_M(v) = \frac{dN}{N\,dv} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, v^2 \qquad (31.12)$$

in this interval can be ignored, that is, the distribution function is assumed to be constant (within the interval). In our case the interval $dv = 1$ m/s is small (compared to the mean speed $\langle v \rangle = \sqrt{8kT/\pi m} \approx 475$ m/s). In addition, the distribution function $f_M(v)$ varies very weakly in the neighborhood of $\langle v \rangle$ (Figure 31.1). Hence, formula (31.11) practically solves the problem. Substituting the value of the mean speed $\langle v \rangle = \sqrt{8kT/\pi m}$ into (31.11), we arrive at the solution to the problem in general form:

$$\frac{dN}{N} = \frac{8\sqrt{2}}{\pi} \left(\frac{m}{\pi kT} \right)^{1/2} e^{-4/\pi} \, dv.$$

Carrying out the necessary calculations, we get

$$dN/N = 1.9 \cdot 10^{-3} = 0.19\%,$$

where we have used the tabulated values of the function $f(x) = e^{-x}$.

Example 31.2. *Find the fractional number of molecules whose speed exceeds the absolute value of the mean velocity.*

Solution. In this case dv is infinite (ranging from $\langle v \rangle$ to ∞) and formula (31.11) cannot be used directly. However, if we integrate (31.11) within the above-noted limits, we find the sought fractional number of molecules:

$$\frac{N_1}{N} = \int_{\langle v \rangle}^{\infty} 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} v^2 \, dv$$

$$= \frac{4}{\sqrt{\pi}} \alpha^{3/2} \int_{\langle v \rangle}^{\infty} e^{-\alpha v^2} v^2 \, dv, \qquad (31.13)$$

where $\alpha = m/2kT$, and N_1 is the number of molecules (with the total number being N) whose speed exceeds the mean speed.

Let us represent the last integral in (31.13) in the following form:

$$\frac{4}{\sqrt{\pi}}\,\alpha^{3/2}\int\limits_{\langle v\rangle}^{\infty} e^{-\alpha v^2}\,v^2\,\mathrm{d}v$$

$$=\frac{4}{\sqrt{\pi}}\,\alpha^{3/2}\int\limits_{0}^{\infty} e^{-\alpha v^2}\,v^2\,\mathrm{d}v-\frac{4}{\sqrt{\pi}}\,\alpha^{3/2}\int\limits_{0}^{\langle v\rangle} e^{-\alpha v^2}\,v^2\,\mathrm{d}v.$$

The first integral on the right-hand side is equal to unity according to the normalization condition (31.2):

$$\frac{4}{\sqrt{\pi}}\,\alpha^{3/2}\int\limits_{0}^{\infty} e^{-\alpha v^2}\,v^2\,\mathrm{d}v=1.$$

To calculate the second integral, we change the variable in it by setting $t = v\sqrt{\alpha}$. Then

$$\frac{4}{\sqrt{\pi}}\,\alpha^{3/2}\int\limits_{0}^{\langle v\rangle} e^{-\alpha v^2}\,v^2\,\mathrm{d}v=\frac{4}{\sqrt{\pi}}\int\limits_{0}^{\sqrt{\alpha}\langle v\rangle} t^2e^{-t^2}\,\mathrm{d}t$$

$$=\frac{4}{\sqrt{\pi}}\int\limits_{0}^{1.13} t^2e^{-t^2}\,\mathrm{d}t, \qquad (31.14)$$

since $\sqrt{\alpha\,\langle v\rangle}=\sqrt{4/\pi}\approx 1.13$. We integrate (31.14) by parts and reduce the last integral to the error function

$$\Phi(z)=\frac{2}{\sqrt{\pi}}\int\limits_{0}^{z} e^{-x^2}\,\mathrm{d}x,$$

whose values are listed in special tables. We obtain

$$\frac{4}{\sqrt{\pi}}\int\limits_{0}^{1.13} t^2e^{-t^2}\,\mathrm{d}t=\frac{4}{\sqrt{\pi}}\left[-t\,\frac{e^{-t^2}}{2}\Bigg|_0^{1.13}+\frac{1}{2}\int\limits_{0}^{1.13} e^{-t^2}\,\mathrm{d}t\right]$$

$$=-0.39+\Phi(1.13).$$

From error-function tables we find that $\Phi(1.13) = 0.89$. Hence the fractional number of molecules whose speed

is greater than the mean is

$$N_1/N = 0.50.$$

Thus, half of the molecules have a speed higher than the mean and half lower than the mean.

Example 31.3. *Find the number of hydrogen molecules that every second cross an area of one square centimeter positioned at right angles to the X axis of a coordinate system (the hydrogen is kept in standard conditions).*

Solution. We give two solutions of this problem.

The first model. Since there is no preferential direction in the motion of the gas molecules, we must assume that one-third of all the molecules fly along the X axis, one-third along the Y axis, and one-third along the Z axis. Hence, one-sixth of the molecules fly in the positive direction of the X axis. We also assume that all the molecules have the same speed equal to $\langle v. \rangle$ Then the sought number of molecules is

$$n_1 = (1/6)\, n_0 \langle v \rangle \, \Delta A \, \Delta t, \tag{31.15}$$

where n_0 is the number of molecules per unit volume, $\Delta A = 1\ \mathrm{cm}^2$ the area through which the molecules fly, and $\Delta t = 1$ s a time interval.

The second model. In the first model all the molecules were assumed to be moving with the same speed $\langle v \rangle$. However, as we know, molecules are distributed in the components of velocity according to the Maxwell distribution, which in the case of one-dimensional motion can easily be obtained from distribution (31.8):

$$\mathrm{d}n(v_x) = n_0 \left(\frac{m}{2\pi kT}\right)^{1/2} \mathrm{e}^{-mv_x^2/2kT}\, \mathrm{d}v_x \tag{31.16}$$

Hence, the number of molecules flying through the area $\Delta A = 1\ \mathrm{cm}^2$ in the course of $\Delta t = 1$ s can be found from the relationship

$$\begin{aligned} n_2 &= \Delta A\ \Delta t \int_0^{\infty} v_x\, \mathrm{d}n(v_x) \\ &= \Delta A\, \Delta t \int_0^{\infty} n_0 \left(\frac{m}{2\pi kT}\right)^{1/2} \mathrm{e}^{-\frac{mv_x^2}{2kT}}\, v_x\, \mathrm{d}v_x \end{aligned}$$

$$= \frac{n_0 \sqrt{\alpha/\pi}}{2\alpha} \Delta A \, \Delta t \int_0^\infty e^{-\alpha v_x^2} \, d(\alpha v_x^2)$$

$$= \Delta A \, \Delta t \, \frac{n_0 \sqrt{\alpha/\pi}}{2\alpha} \int_0^\infty e^{-t} \, dt$$

$$= \Delta A \, \Delta t \, \frac{n_0}{2\sqrt{\alpha\pi}} [-e^{-t}]\big|_0^\infty = \frac{n_0 \, \Delta A \, \Delta t}{2\sqrt{\alpha\pi}},$$

with $\alpha = m/2kT$. Since $\langle v \rangle = \sqrt{8kT/\pi m}$, we have

$$n_2 = (1/4)\, n_0 \langle v \rangle \, \Delta A \, \Delta t. \tag{31.17}$$

We see that the expressions (31.15) and (31.17) differ considerably. Carrying out the necessary calculations

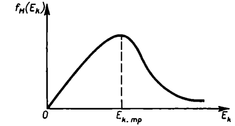

Figure 31.2

and allowing for the fact that $n_0 = p_0/kT_0$, where p_0 and T_0 are the standard pressure and temperature, we get

$$n_1 \approx 7.4 \times 10^{23}, \quad n_2 \approx 11.1 \times 10^{23}.$$

Example 31.4. *A vessel of volume* $V = 30$ l *contains* $m = 100$ g *of oxygen under a pressure* $p = 3 \times 10^5$ Pa. *Determine the most probable value of the kinetic energy of the oxygen molecules.*

Solution. One can easily show that oxygen in the specified conditions constitutes an ideal gas. The most probable value of the kinetic energy of the oxygen molecules corresponds to the maximum in the Maxwell distribution (31.10) in kinetic energy (Figure 31.2) and can be found

from the appropriate distribution function,

$$f_M(E_k) = 2\pi \left(\frac{1}{\pi kT}\right)^{3/2} E_k^{1/2} e^{-E_k/kT}. \qquad (31.18)$$

Hence, the problem is reduced to finding the extremum of function (31.18). Finding the first derivative of $f'_M(E_k)$ and nullifying it, we get

$$f'_M(E_k) = 2\pi \left(\frac{1}{\pi kT}\right)^{3/2} e^{-E_k/kT} \left(-\frac{1}{kT}\right) E_k^{1/2} + 2\pi \left(\frac{1}{\pi kT}\right)^{3/2} \frac{1}{2} E_k^{-1/2} e^{-E_k/kT} = 0.$$

Hence, the most probable kinetic energy of the molecules is

$$R_{k\ mp} = kT/2. \qquad (31.19)$$

The temperature can be found from the ideal gas law

$$T = pVM/mR.$$

Using (31.18) and (31.3), we can find the mean value of the kinetic energy of the molecules (in translational motion):

$$\langle E_k \rangle = \frac{3}{2} kT.$$

Thus, the mean kinetic energy of molecules of an ideal gas is three times the most probable value of the kinetic energy:

$$\langle E_k \rangle / E_{k\ mp} = 3.$$

Note that the ratio of the average speed of molecules and the most probable speed is lower than three:

$$\frac{\langle v \rangle}{v_{mp}} = \frac{\sqrt{8kT/\pi m}}{\sqrt{2kT/m}} \approx 1.13.$$

32. The Boltzmann Distribution

The Boltzmann distribution (31.6) for the one-dimensional case assumes the form

$$dw(x) = \frac{dN(x)}{N} = B_1 e^{-U(x)/kT}\, dx, \qquad (32.1)$$

where $dN(x)$ is the number of particles contained in a layer of thickness dx in the neighborhood of coordinate x (N is the total number of particles). Let us apply distribution (32.1) to the atmosphere of Earth. We assume that the temperature of air in Earth's atmosphere is constant ($T = \text{const}$; the case of an isothermal atmosphere) and that the altitude h to which the atmosphere extends is much smaller than Earth's radius R ($h \ll R$), which means that the acceleration of free fall in the atmospheric layer is constant ($g = 9.8\ \text{m/s}^2 = \text{const}$). Then the potential energy of a molecule of mass m at an altitude x above Earth's surface is $U(x) = mgx$.

Using the normalization condition (31.2), we can find the value of constant B_1:

$$1 = \int_0^\infty B_1 e^{-mgx/kT}\, dx, \quad B_1 = mg/kT.$$

Thus, the number of molecules $dN(x)$ in a layer of air of thickness dx at an altitude x above the surface of Earth is

$$dN(x) = \frac{Nmg}{kT} e^{-mgx/kT}\, dx. \tag{32.2}$$

Suppose that dA is the elementary area perpendicular to the X axis. Then $dx\, dA = dV$ is the volume element, and

$$\frac{Nmg}{dA} = p_0$$

is the pressure of the atmosphere at the surface of Earth. Hence, the number of molecules contained in a volume dV at an altitude x is

$$dN(x) = \frac{p_0}{kT} e^{-mgx/kT}\, dV.$$

Since $dN(x)/dV = n$ is the number of molecules per unit volume at altitude x and $p_0/kT = n_0$ is the same quantity in the surface of Earth, we have

$$n = n_0 e^{-mgx/kT}. \tag{32.3}$$

This yields the following *barometric height formula*

$$p = p_0 e^{-mgx/kT}.$$

Standard problems involving the Boltzmann distribution, like those involving the Maxwell distribution, are reduced to finding the means of physical quantities and the number of particles possessing specific properties.

Example 32.1. *Find the mean potential energy of air molecules in Earth's gravitational field. At what altitude above the surface of Earth is the energy of the molecules equal to the mean potential energy? The temperature of the air is assumed constant and equal to* 0 °C.

Solution. The gas (air) is in Earth's gravitational field. Hence, its molecules are distributed in energy according to the Boltzmann distribution:

$$f_B(U) = Be^{-U/kT},$$

where $U = mgh$ is the potential energy of a molecule.

If we know the distribution function f for the molecules in a specific physical parameter l (speed v, momentum p, energy E, etc.), the mean value of a specific physical quantity that is a function of this parameter, $\varphi = \varphi(l)$, is determined from the expression

$$\langle \varphi(l) \rangle = \frac{\int_0^\infty \varphi(l) f(l) \, dl}{\int_0^\infty f(l) \, dl}.$$

In our case $\varphi(l) = U$ and $f = f_B$. Thus, the mean value of the potential energy of air molecules in Earth's gravitational field is

$$\langle U \rangle = \frac{\int_0^\infty BUe^{-U/kT} \, dU}{\int_0^\infty Be^{-U/kT} \, dU} = \frac{\int_0^\infty Ue^{-U/kT} \, dU}{\int_0^\infty e^{-U/kT} \, dU}. \quad (32.4)$$

Here the denominator

$$\int_0^\infty e^{-U/kT}\,dU = kT$$

and the numerator

$$\int_0^\infty U e^{-U/kT}\,dU = k^2T^2 \int_0^\infty t e^{-t}\,dt = k^2T^2.$$

Hence, $\langle U \rangle = kT$.

Now let us find the altitude h at which the potential energy of the air molecules is equal to the mean potential energy: $\langle U \rangle = mgh$, or $kT = mgh$. Hence,

$$h = \frac{kT}{mg} = \frac{RT}{Mg}, \quad h \approx 8 \times 10^3 \text{ m}.$$

Example 32.2. *Determine the mass of air contained by a cylinder with base area $\Delta A = 1$ m^2 and altitude $h =$ 1 km. Assume that the air is in standard conditions.*

Solution. Here we cannot apply the ideal gas law since the physical system, the ideal gas (air), is in Earth's gravitational field. We cannot directly employ formula (32.2) either because the layer thickness $dx = h = 1$ km is large. Integrating (32.2) with respect to x from 0 to h, we can find the total number of air molecules in the cylinder:

$$N_1 = \frac{Nmg}{kT} \int_0^h e^{-\frac{mgx}{kT}}\,dx = N\left(1 - e^{-\frac{mgh}{kT}}\right)$$

$$= \frac{p_0\,\Delta A}{mg}\left(1 - e^{-\frac{mgh}{kT}}\right).$$

Multiplying (32.5) by the mass m of one molecule, we arrive at the expression for the sought mass:

$$M_1 = mN_1 = \frac{p_0\,\Delta A}{g}\left(1 - e^{-\frac{mgh}{kT}}\right) - \frac{p_0\,\Delta A}{g}\left(1 - e^{-\frac{Mgh}{RT}}\right)$$

where $M = 29$ kg/mol is the molar mass of air.

If the air did not experience a gravitational pull, then the ideal gas law would yield

$$M_2 = p_0 \Delta A h M / RT.$$

Let us find the M_2 to M_1 ratio:

$$\alpha = \frac{M_2}{M_1} = \frac{Mgh}{RT\,(1 - e^{-Mgh/RT})}.$$

A numerical calculation for altitudes $h_1 = 100$ m, $h_2 = 1$ km, and $h_3 = 10$ km yields the following values for this ratio: $\alpha_1 = 1.008$, $\alpha_2 = 1.08$, and $\alpha_3 = 1.8$.

We see that for air contained in a volume with an altitude of several hundred meters we can apply the ideal gas law and ignore the Boltzmann distribution. For air contained in a volume with an altitude of one kilometer and more the use of the ideal gas law leads to considerable errors and in such cases we must allow for Earth's gravitational field.

It would be interesting to study the dependence of α on parameters (the temperature T, the molar mass M, and the acceleration of free fall g).

Example 32.3. *The atmosphere contains dust particles with a particle mass* $m = 8 \times 10^{-22}$ kg *and a particle volume* $V = 5 \times 10^{-22}$ m^3. *Find the decrease in the concentration of these particles at altitudes* $h_1 = 3$ m *and* $h_2 = 30$ m. *The air is in standard conditions.*

Solution. The physical system consists of dust particles and air molecules in the Earth's gravitational field. Hence, both the dust particles and the air molecules obey the Boltzmann distribution (32.3). For air molecules this distribution can be applied directly, while for dust particles doing so may lead to considerable errors. The fact is that in addition to being subjected to the force of gravity mg, the dust particles are acted upon by a buoyancy force F_b since they are submerged in air. A simple calculation shows that F_b is comparable in order of magnitude with the force of gravity mg. Indeed, the density of a dust particle is $\rho_d = m/V$, or $\rho_d = 8 \times 10^{-22}/5 \times 10^{-22} \approx 1.6$ kg/m^3, which differs little from the air density $\rho_{air} \approx 1.3$ kg/m^3. This means that the

buoyancy force differs little from the force of gravity. For this reason we first find the effective mass of a dust particle, $m_{eff} = g - F_b/g$, or $m_{eff} = m - \rho_{air}V$, where ρ_{air} is the air density. The air density can be found from the ideal gas law:

$$\rho_{air} = pM/RT.$$

Hence,

$$m_{eff} = m - \frac{pMV}{RT}.$$

Employing the Boltzmann density (32.3), we find the variation of the concentration of dust particles with altitude:

$$\beta = \frac{n}{n_0} = e^{-\frac{m_{eff}gh}{kT}} = e^{-\frac{(m - pMV/RT)gh}{kT}}.$$

A numerical calculation for $h_1 = 3$ m and $h_2 = 30$ m yields $\beta_1 = 0.29$ and $\beta_2 = 3 \times 10^{-6}$, respectively. Thus, while at the altitude $h_1 = 3$ m (the height of a one-story building) the dust concentration is still approximately one-third of the dust concentration at the surface of Earth, at an altitude $h_2 = 30$ m (the top of a ten-story building) there is practically no dust. This conclusion is valid if there is no updrift.

Part 3

SOLUTION OF NONSTANDARD, NONSPECIFIED, AND ARBITRARY PROBLEMS

Chapter 12

NONSTANDARD AND ORIGINAL PROBLEMS

33. Nonstandard Problems

Earlier we noted that concrete and generalized knowledge is not sufficient to solve nonstandard problems. As a rule when we apply such knowledge to problems of this type, we end up with an open system of equations. Then we have to look for an unspecified "something" that will enable us to close the system of equations. In nonstandard problems this unspecified "something" is so diversified that any attempt to classify such problems proves futile. In the few examples considered below we will specify the characteristic features of this "something" and ways of establishing them.

Example 33.1. *An object of mass m is lying on a horizontal surface, with a coefficient f of friction between surface and object. At time $t = 0$ a horizontal force varying according to the law $\mathbf{F} = t\mathbf{a}$, with $\mathbf{a}$ a constant vector, is applied to the object. Find the length of the path traveled by the object after the first t seconds have elapsed.*

Solution. We carry out an ordinary analysis of the problem by applying the method of analyzing its physical content. Only the object of mass m constitutes the physical system, and we consider this object a particle. Several forces influence the motion of the body, and one depends on time. We must find one of the parameters of this motion, the path traveled by the object. This constitutes a basic problem of particle mechanics.

Let us apply Newton's second law. We link an inertial reference frame with the horizontal surface and direct the X axis along vector **a**. Four forces act on the object during the motion: the given force $\mathbf{F} = \mathbf{a}t = \mathbf{i}at$, where **i** is the unit vector pointing in the positive direction of the X axis, the force of friction $\mathbf{F}_{fr} = -fmg\mathbf{i}$, the force **N** by the support (the horizontal surface) on the object, and the force of gravity $m\mathbf{g}$. The last two forces compensate each other. By Newton's second law,

$$m\frac{d\mathbf{v}}{dt} = at\mathbf{i} - fmg\mathbf{i},$$

or in terms of projection on the X axis,

$$m\frac{dv_x}{dt} = at - fmg.$$

Integrating this equation and allowing for the initial conditions, we arrive at the law of variation of velocity:

$$v_x = \frac{at^2}{2m} - fgt.$$

Integrating the equation $dx/dt = v_x$, we obtain the law of motion:

$$x = \frac{at^3}{6m} - \frac{fgt^2}{2}. \qquad (33.1)$$

The last expression provides the answer to the problem. However, the solution is invalid. The physical analysis has been carried out formally, that is, we did not allow for static friction and, especially, for the fact that this force, just as the given force $\mathbf{F} = \mathbf{a}t$, varies (grows) with time. The object starts moving only at time $t_0 = fmg/a$, when the static friction reaches its maximum value. Prior to this moment the object was at rest. Substituting into Eq. (33.1) $t - t_0$ for t, we arrive at the correct equation:

$$x = \frac{a(t-t_0)^3}{6m} - \frac{fg(t-t_0)^2}{2},$$

where $t \geqslant t_0$.

Thus, in the given nonstandard problem the "something" was a thorough analysis of the force of friction

and the fact that this force varies with time, specifically, grows from zero to its maximum value fmg as the given force $\mathbf{F} = \mathbf{a}t$ grows.

A condition very often encountered in nonstandard problems is the *separation condition*; when the interaction of objects ceases, the elastic force exerted by the support vanishes, or $N = 0$.

Example 33.2. *A time-dependent force $F = at$, where a is constant, starts to act at time $t = 0$ on a small object of mass m lying on a smooth horizontal surface. The forc-always forms an angle α with the horizontal surface. Find the moment in time when the object is separated from the surface and the velocity of the object before and after separation (lift-off).*

Solution. The physical system consists of only one body, object m. All other bodies are considered external. The given object may be considered a particle. As a result of the interaction with external bodies the given object moves. Note that one of the external forces depends on time t. We wish to find the moment when a certain event takes place (lift-off) and the velocity of the object before and after the event. Since the motion of the object is not considered formally (the force is specified), the given example is related to a basic problem of particle mechanics.

Let us employ Newton's second law. We link an inertial reference frame with the horizontal plane, direct the X axis along the plane, and direct the Y axis vertically upward. By Newton's second law,

$$m \frac{dv_x}{dt} = at \cos \alpha, \tag{33.2}$$

$$m \frac{dv_y}{dt} = N + at \sin \alpha - mg. \tag{33.3}$$

It is easy to guess (!) that prior to lift-off $v_y = \text{const} = 0$ and, hence, the system of equations (33.2), (33.3) assumes the form

$$m \frac{dv_x}{dt} = at \cos \alpha, \tag{33.4}$$

$$0 = N + at \sin \alpha - mg. \tag{33.5}$$

We have arrived at a system of two equations in three unknowns (v_x, N, and t), but if we guess (!) that at the moment of lift-off the force exerted by the support on the object vanishes ($N = 0$), then Eq. (33.5) immediately gives us that moment:

$$t_0 = \frac{mg}{a \sin \alpha}. \tag{33.6}$$

Next, integrating Eq. (33.4) and allowing for the initial conditions, we obtain the law of variation of velocity prior to lift-off:

$$v_x = \frac{at^2 \cos \alpha}{2m}. \tag{33.7}$$

After lift-off ($N = 0$) the system of equations (33.2), (33.3) assumes the form

$$m \frac{dv_x}{dt} = at \cos \alpha, \tag{33.8}$$

$$m \frac{dv_y}{d(t-t_0)} = a(t-t_0) \sin \alpha - mg. \tag{33.9}$$

In Eq. (33.9) we have allowed (!) for the fact that the motion along the Y axis starts at time of lift-off $t_0 = mg/a \sin \alpha$. Integrating Eqs. (33.8) and (33.9) and allowing for the initial conditions, we arrive at the law of variation of the velocity after lift-off ($t \geqslant t_0$):

$$\mathbf{v} = \frac{at^2 \cos \alpha}{2m} \mathbf{i} + \left[\frac{a(t-t_0)^2 \sin \alpha}{2m} - g(t-t_0) \right] \mathbf{j}, \tag{33.10}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors directed along the X and Y axes.

At first it may seem that the guesses made in the process of solving the problem (prior to lift-off $v_y = 0$, at the moment of lift-off N vanishes, and after lift-off the time it takes the object to move along the Y axis is $t - t_0$) are minute, inessential. Indeed, they are, but without making them we cannot solve the problem. This illustrates the important role played by the little "something." After solving such problems our experience and physical intuition become richer. Gradually these details, or guesses, do indeed become obvious. But everything that has been mastered seems simple and obvious, while

the unknown and unsolved seems complicated and incomprehensible.

Another widely used tool in solving nonstandard problems is the *choice of a simple reference frame* (RF). Here are three examples. In the first the choice of the reference frame makes no difference, in the second it is so important that one choice makes the problem standard while another makes it nonstandard, and in the third a successful choice of RF becomes a decisive factor (a common standard problem becomes an original one).

Example 33.3. *A chain of mass m is laid out in a circle of radius R and fitted onto a smooth circular cone whose semi-vertex angle is* θ *(Figure 33.1). Find the tensile stress*

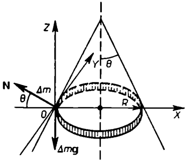

Figure 33.1

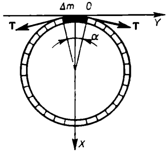

Figure 33.2

developed by the chain if the cone (with the chain) rotates with a constant angular velocity ω *about the vertical axis, which coincides with the cone's symmetry axis.*

Solution. The physical system consists of a single object, the chain. The chain cannot be thought of as being a particle. The sought tensile stress acts on each element of the chain. Hence, we partition the chain into similar elements so small that each may be assumed to be a particle. We consider one such element (or particle) of mass Δm. It moves along a circle of known radius R with a known angular velocity ω, and certain forces

act on it. We must find one of these forces. This constitutes one of the basic problems of the dynamics of rotational motion of a particle.

Let us employ Newton's second law. It can easily be shown that the choice of a reference frame (inertial or noninertial) is irrelevant; we will write the system of equations for this problem simultaneously in two reference frames. Let us select the noninertial reference frame (NIRF) as the one linked to element Δm and the inertial reference frame (IRF) as the one linked to any (external) fixed object. Four forces act on Δm in IRF: the force of gravity Δmg (Figure 33.1), the elastic force N exerted by the support (cone) on the element, and two equal (why?) tensile stresses $\mathbf{T}$ (Figure 33.2), each of which is directed along the tangent to the circle at the appropriate point. In the NIRF there is an additional fifth force, the centrifugal force of inertia $\Delta m\omega^2 R$. Projecting the forces on the coordinate axes, we can write Newton's second law in the following form for the IRF:

$$N \sin\theta - 2T \sin(\alpha/2) = 0 \quad \text{(for the } Z \text{ axis)}, \quad (33.11)$$

$$2T \sin(\alpha/2) - N \cos\theta = \Delta m\omega^2 R \quad \text{(for the } X \text{ axis)}. \quad (33.12)$$

Correspondingly, for the NIRF:

$$N \sin\theta - 2T \sin(\alpha/2) = 0, \quad (33.13)$$

$$2T \sin(\alpha/2) - N \cos\theta - \Delta m\omega^2 R = 0. \quad (33.14)$$

Obviously, the system of equations (33.11), (33.12) is equivalent to the system (33.13), (33.14), but in either case it constitutes an open system of two equations in three unknowns (T, α, and N). Hence, we must find the "something" that will close the system of equations. Looking at Figure 33.2, we guess that angle α is in some way connected with the element Δm. It is easy to see (why?) that this connection has the form

$$\Delta m/m = \alpha/2\pi.$$

Finally, if we allow for another "something," the fact that angle α is small (and, therefore, $\sin\alpha \approx \alpha$), we arrive at a simple, closed system of equations (in the

IRF)

$$N \sin\theta - \Delta mg = 0,$$
$$T\alpha - N\cos\theta = \Delta m\omega^2 R,$$
$$\Delta m/m = \alpha/2\pi.$$

After solving this system we arrive at the answer in general form:

$$T = \frac{m(\omega^2 R + g\cot\theta)}{2\pi}.$$

Note that although concrete and generalized knowledge plays an essential part in the solution of this problem, the role of the insignificant "something" has increased.

Example 33.4. *A massless pulley is attached to the ceiling of an elevator, then a massless string is swung over the pulley, and the ends of the string are tied to two loads of mass m_1 and m_2 Figure (33.3). The elevator is lifted with an acceleration* **a**. *Ignoring friction, find the force with which the pulley acts on the ceiling of the elevator.*

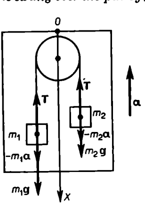

Figure 33.3

Solution. We will solve the problem in two reference frames, one linked with the elevator cabin (NIRF) and the other linked with Earth (IRF).

Solution in NIRF. The physical system incorporates the two loads m_1 and m_2 (which can be thought of as particles), the massless pulley, and the massless string. The objects of the system are in accelerated motion due to the action of certain forces. We must find the force with which the pulley acts on the ceiling of the elevator cabin. This constitutes a basic problem of dynamics.

Let us apply Newton's second law to m_1 and m_2 in the selected NIRF. We send the X axis downward (see Figure 33.3). Three forces act on object m_1: the force of gravity $m_1\mathbf{g}$, the tensile stress $\mathbf{T}$ developed by the string,

and the force of inertia $-m_1\mathbf{a}$. Projecting these forces on the X axis, we obtain the expression for Newton's second law as applied to m_1:

$$m_1 g + m_1 a - T = m_1 b,$$

where b is the projection of the acceleration of m_1 in the selected NIRF. Here we have assumed that $m_1 > m_2$, so that $\mathbf{b}$ is directed downward. Correspondingly, for m_2 we have

$$m_2 g + m_2 a - T = -m_2 b.$$

We have a closed system of two equations in two unknowns (T and b). The sought force F can be found by writing Newton's second law for the pulley's center of mass (which is fixed in the NIRF): $2T - F = 0$. Hence,

$$F = \frac{4m_1 m_2 (g+a)}{m_1 + m_2}. \qquad (33.15)$$

Thus, in the selected NIRF the problem has proved to be standard.

Solution in IRF. Newton's second law as applied to m_1 and m_2 yields

$$m_1 g - T = m_1 a_1, \qquad (33.16)$$

$$m_2 g - T = -m_2 a_2, \qquad (33.17)$$

where a_1 and a_2 are absolute values of the projection of the accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ of objects m_1 and m_2, respectively, in the selected IRF. We have an open system of two equations in three unknowns (T, a_1, and a_2). Hence, we must find the "something" that will close the system.

We guess that a_1 and a_2 are related. But how? Obviously (!),

$$a_2 = a_1 \quad 2a. \qquad (33.18)$$

Indeed, since $a_2 = b + a$ and $a_1 = b - a$, we have

$$a_2 - a_1 = 2a.$$

Solving the closed systems of equations (33.16)-(33.18), we find that

$$T = \frac{2m_1 m_2 (g+a)}{m_1 + m_2}.$$

Hence,

$$F = 2T = \frac{4m_1m_2(g+a)}{m_1+m_2},$$

which coincides with (33.15). Thus, the same problem in the specified IRF has proved to be nonstandard.

Example 33.5. *The projectile of an antiaircraft gun is fired upward with muzzle velocity* $\mathbf{v}_0$ *and explodes at the highest point of its trajectory into n equal parts. The fragments have equal velocities,* $\mathbf{u}_0$, *directed at different polar* (θ) *and azimuthal* (φ) *angles. Determine the position of an arbitrary fragment at any moment in time.*

Solution. We select an inertial reference frame, the one linked with Earth. It is easy to see that with such a choice

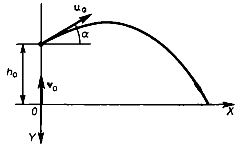

Figure 33.4

the problem is standard. The physical system consists of only one fragment, which can be thought of as being a particle. The particle's initial altitude $h_0 = v_0^2/2g$, the initial velocity $\mathbf{u}_0$, and the angle $\alpha = 90° - \theta$ that $\mathbf{u}_0$ makes with the horizon (Figure 33.4) are known. We must find the position of the fragment at any moment of time t. This constitutes a basic problem of particle kinematics.

Let us select the plane in which an arbitrary fragment moves (along a parabola) as the coordinate plane OXY (Figure 33.4). Then the law of motion of the fragment in

parametric form has the form

$$x = u_0 \cos \alpha t, \quad y = -\frac{v_0^2}{2g} \pm u_0 \sin \alpha t + \frac{gt^2}{2}.$$

Obviously, this law of motion makes it possible to find the position of the fragment at any moment in time.

Now we take the reference frame linked with the center of mass of the projectile as the NIRF. After the explosion the center of mass (the origin of the NIRF) moves downward with an acceleration g. In this reference frame the motion of any fragment is uniform (with a velocity u_0) and, therefore, at every moment all fragments form a sphere of radius $R = u_0 t$ with the center at the origin. We have solved the problem, and the solution is not only simple and elegant but very graphic.

Thus, the same problem considered in a specified NIRF has proved to be an original problem.

34. Original Problems

It is better to speak not of standard, nonstandard, or original problems (since we have just seen that the same problem can be considered a standard, nonstandard, or even original depending, for one thing, on the choice of reference frame) but of ways of solving problems (standard, nonstandard, and original).

Trying to classify original problems is, obviously, just as meaningless as trying to classify nonstandard problems in general. It can only be noted that original problems often allow for a standard, nonstandard, or original solution. In the first case it is sufficient to employ only concrete and generalized knowledge, in the second one often resorts to guesses (though this plays no essential role in the solution process), and, finally, in the third the problem can be solved only by intuition and a wild guess. It is problems of the third group that we can call original. Here are some examples.

Example 34.1. *Out of a uniform solid circle of radius R we cut a circle of radius $r < R/2$ centered at a distance $a < (R - r)$ from the center of the larger circle (Fig-*

ure 34.1a). Find the position of the center of mass of the resulting figure.

Solution. The problem allows for a standard solution, following from the definition of the center of mass (see Eq. (13.2)). But the standard solution involves rather cumbersome computations. Let us try a nonstandard, or maybe even an original, solution. What special feature characterizes the given physical system? As is known,

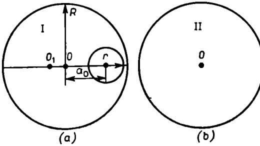

Figure 34.1

circles are symmetric figures with an infinitude of symmetry axes (the diameters of the circles). The center of mass of the larger circle (we denote this figure by the Roman numeral II in Figure 34.1*b*) lies (before the smaller circle was cut out) at the center of the circle (point O). The figure left after we cut out the smaller circle (we denote this figure by the Roman numeral I) is still symmetric, but possesses only one axis of symmetry, OO_1.

But what if we were to cut out two small circles instead of one in such a way that the two are positioned symmetrically in relation to the larger circle (Figure 34.2*a*). The center of mass of the resulting figure (we denote this figure by III) will be positioned at point O (this follows from symmetry considerations). Now we return the second small circle (the left one) to its place (the resulting figure is denoted by IV in Figure 34.2*b*). Then the problem of finding the center of mass of figure I is reduced to finding the center of mass of the system of figures III

and IV, whose centers of mass are known. Since the center of mass of two objects lies on the straight line connecting their centers of mass at a point A (Figure 34.2a) that

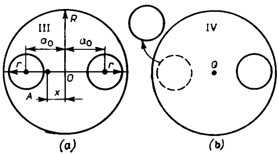

Figure 34.2

divides the distance a_0 between them in a ratio inversely proportional to their masses, we have

$$\frac{x}{a_0 - x} = \frac{m_1}{m_4}, \tag{34.1}$$

where $x = |AO|$, $m_1 = \alpha\pi r^2$ is the mass of figure IV, and $m_4 = \alpha(\pi R^2 - 2\pi r^2)$ the mass of figure III, with $\alpha = \text{const}$. From (34.1) we find the sought position of the center of mass of figure I:

$$x = \frac{a_0 r^2}{R^2 - r^2}.$$

Note that the most important point in the second (original) solution of this problem was the guess about cutting out an additional small circle. This guess constitutes an element of experience, of physical intuition.

Let us make the terms of the problem more complicated. Suppose that we are dealing with an asymmetric figure, say a triangle, out of which we have cut out a circle.

Example 34.2. *Out of a triangular plate with sides a, b, and c we cut a circle of radius r centered on a median AD*

of the triangle at a distance $a_1 = |MN|$ from the point M where the medians intersect (Figure 34.3). Find the position of the center of mass of the resulting figure.

Solution. It is easy to see that this problem allows for a standard solution via formula (13.2) involving cumbersome calculations. Let us try another approach. The method of cutting out a second circle so as to obtain a symmetric figure does not work here because the initial

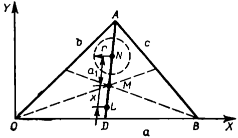

Figure 34.3

figure, the triangle, is not symmetric. But the idea of combining various figures can still be used. Let us return the small circle to its proper place (we denote this small circle by I). Then we have a triangle OAB (we denote it by III) whose center of mass lies at the point M where all three medians intersect. But the center of mass of figure II (the triangular plate with the circle cut out) lies at point L on median AD at an unknown distance x from point M, and we guess that figure III may be considered as the combination of figures I and II. Since the mass of each figure is known ($m_1 = \alpha\pi r^2$, $m_2 = m_3 - m_1$, and $m_2 = \alpha\ [\sqrt{p(p-a)(p-b)(p-c)} - \pi r^2]$, where $p = (a+b+c)/2$), Eq. (34.1), which gives the center of mass of figures I and II, assumes the form

$$x/a_1 = m_1/m_2.$$

In general form the solution is

$$x = \frac{\pi r^2 a_1}{\sqrt{p(p-a)(p-b)(p-c)} - \pi r^2}.$$

Example 34.3. *A uniformly charged ball has a spherical cavity whose center lies at a distance a from the ball's center (Figure 34.4). Find the electric field strength at an arbitrary point inside the cavity if the charge density is* ρ.

Solution. The physical system consists of a uniformly charged ball with a cavity. We need to calculate the electric field inside the cavity. This constitutes a basic problem of the theory of an electrostatic field.

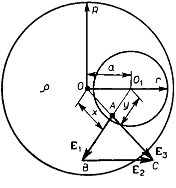

Figure 34.4

Since the charge of the ball cannot be considered point-like, the DI method could be employed (see Section 6), but this method involves time-consuming evaluation of integrals. Let us instead employ the combination ideas discussed above. We denote the radius of the cavity by r and that of the ball by R. We consider a collection of three objects uniformly charged with electricity with a charge density ρ: a small ball of radius r (denoted by I), a large ball of radius R (denoted by III), and the ball with the cavity (denoted by II). According to the superposition principle, the electric field strength $\mathbf{E}_3$ at any point inside the big ball is equal to the vector sum of the electric field strengths of the small ball, $\mathbf{E}_1$, and the ball with the cavity, $\mathbf{E}_2$:

$$\mathbf{E}_3 = \mathbf{E}_1 + \mathbf{E}_2.$$

Hence, the sought field strength is

$$\mathbf{E}_2 = \mathbf{E}_3 - \mathbf{E}_1.$$

Suppose that an arbitrary point A inside the cavity is positioned at a distance y from the center of the cavity and at a distance x from the center of the ball (Figure 34.4). Then the formula for the yield strength inside a charged

ball yields

$$E_1 = \rho y/3\varepsilon_0, \tag{34.2}$$

$$E_3 = \rho x/3\varepsilon_0. \tag{34.3}$$

Let us consider triangles AOO_1 and ABC. Allowing for (34.2) and (34.3), we find that

$$\frac{|AB|}{|AC|} = \frac{E_1}{E_3} = \frac{y}{x} = \frac{|AO_1|}{|AO|}.$$

Hence, these triangles are similar, whereby

$$\frac{|BC|}{|OO_1|} = \frac{|AC|}{|AO|}, \quad \text{or} \quad \frac{E_2}{a} = \frac{E_3}{x}.$$

Thus,

$$E_2 = a\rho/3\varepsilon_0.$$

Since $\mathbf{E}_2$ is parallel to OO_1 (this follows from the similarity of triangles AOO_1 and BAC), we finally conclude that the electric field inside the cavity is homogeneous.

Example 34.4. *A direct constant current of density j is flowing in an infinitely long cylindrical conductor. The conductor contains an infinitely long cylindrical cavity whose axis is parallel to that of the conductor and is a distant from it (Figure 34.5). Determine the magnetic field strength at an arbitrary point inside the cavity.*

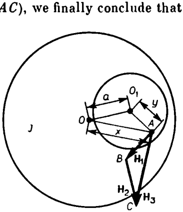

Figure 34.5

Solution. Let us employ the method developed above. We consider an arbitrary point inside the cavity, denoted by A, that is x distant from the axis of the big cylinder and y distant from that of the cylindrical cavity (Figure 34.5). It is expedient to consider three systems: the current and its magnetic field inside a solid (i.e.

having no cavity) infinitely long cylinder (we denote it by III), the current and its magnetic field inside the cylinder with the cavity (we denote it by II), and the current and its magnetic field inside a small (but infinitely long) cylinder whose radius is equal to that of the cavity (we denote it by I). Since by the superposition principle

$$\mathbf{H}_3 = \mathbf{H}_1 + \mathbf{H}_2,$$

we find that the sought magnetic field strength is

$$\mathbf{H}_2 = \mathbf{H}_3 - \mathbf{H}_1.$$

From the theorem on the circulation of vector $\mathbf{H}$,

$$H_1 = \frac{j}{2}\, y, \quad H_3 = \frac{j}{2}\, x.$$

Hence,

$$\frac{H_1}{H_3} = \frac{y}{x} = \frac{|AO_1|}{|AO|}.$$

Since the angles BAC and OAO_1 are equal, the triangles ABC and OAO_1 are similar. Thus,

$$H_2/a = H_3/x, \quad \text{or} \quad H_2 = ja/2.$$

It can easily be shown that $\mathbf{H}_2$ is perpendicular to OO_1; hence, the magnetic field inside the cavity is homogeneous.

Note that solution of the last problem proved to be standard because prior to the problem we solved three almost similar problems and our experience and physical intuition grew with each one. We are now able to formulate dozens of similar problems, and they will be standard rather than original because in the process of solving the first three problems we found a special method for their solution. Thus, the concepts of standard, nonstandard and original problems are very arbitrary and relative and depend on the experience and physical intuition of the person solving the particular problem. Nevertheless, it is useful to classify problems as standard, nonstandard, or original.

Chapter 13
NONSPECIFIED, RESEARCH, AND ARBITRARY PROBLEMS

35. Nonspecified Problems

In Section 8 we mentioned nonspecified problems in connection with the problem statement method. Here we discuss such problems in detail.

The reader will recall that we defined a nonspecified problem as one with an incomplete system of data for its solution or a nonidealized one or a problem with both features present. The solution of a nonspecified problem begins with specifying the problem:

> before solving a problem, formulate it.

However, presenting the general solution of a nonspecified problem as two consecutive steps (statement and solution) reflects only the external aspect of solving a nonspecified problem. For a general solution one must see not only the interrelation and permeation of these steps but also their order of succession. The very process of formulating the first problem (usually the simplest) prepares its solution. The subsequent process of analyzing the first problem prepares the conditions for making the necessary step in solving a more complicated problem, and so on. Here we have the inner dialectics of the process of finding a general solution for a nonspecified problem.

Nevertheless, the most important, decisive step is to formulate the problem. There is good reason to say that

> once a problem has been properly formulated, half of the work of solving it has been done.

Formulation of a problem (as well as solution of a specified problem) begins with the choice of a physical system. One must establish which objects are included in a given system and which will be considered external. Then the physical system is analyzed, that is, first, one establishes what properties the objects in the system possess (classical or quantum, elastic or inelastic, rigid or otherwise, etc.) and, second, in what conditions the objects in the system operate. As is known, the properties of the

objects of a system and the conditions in which they operate determine the physical phenomena occurring in the system. A physics problem always emerges inside a physical phenomenon and reflects the phenomenon.

The process of analyzing a physical system begins with *idealization of the problem*: we introduce ideal objects (particles, massless objects, point charges, etc.) and estimate interactions and interrelations (determine what interactions can be ignored, etc.). Idealization of a problem is carried out practically to the very end of the solution of a nonspecified problem, and here it is important to distinguish two interconnected and interwoven processes: simplification of the terms of the problem and their complication. At the beginning the first process is predominant. Starting the solution, one must introduce as many simplifying assumptions as possible, ignore some properties of the objects, disregard various conditions and the like. However, as the problem gets more complicated (in other words, as we ignore fewer and fewer conditions), the importance of the second process, complication, grows, although here too one may introduce some simplifying assumptions and limitations.

After the physical system has been chosen and analyzed, we begin to analyze the physical phenomena that may occur in the system under certain conditions. At this stage, too, idealization of the problem continues. Various ideal processes are introduced and studied, further simplification of conditions occurs, possible limitations are studied, and so on. It is at this stage, that is, during the analysis of a physical phenomenon, that the first problem emerges: certain data and the sought quantities are selected and the terms of the problem are formulated. Now the problem has been specified. Whether the formulation is meaningful or not will be apparent only when the problem has been solved in general form. Only then will it be clear whether all the data for obtaining a numerical answer are present. If some quantities prove to be unknown, their values must be added to the original conditions. After the solution of the first problem has been analyzed, the terms are made more complicated, and he second problem is formulated. This process continues.

Thus, as noted earlier, to each nonspecified problem there can be assigned a whole cluster of problems of varying difficulty.

Nonspecified problems are so diverse that it is not always possible to apply the above scheme to their solution. Indeed, there simply cannot be any rigid, unified scheme of this sort because solving a nonspecified problem is a creative act. But no matter what nonspecified problem we may be solving, it is impossible to bypass the formulating process and, hence, the idealization of the problem. Below we give an example of a nonspecified problem that at first glance seems to be of a very general nature.

Example 35.1. *Study the motion of two electrically charged objects.*

Solution. The terms of the problem do not state what is known and what must be found, in short, the problem is nonspecified. Let us start with the first step, the statement of the problem. The physical system consists of the two given objects and Earth (we assume that all physical phenomena involving the objects occur on Earth). The effect of all other external objects will be ignored. We know that at this stage we must idealize the problem, and, hence, must introduce and allow for various simplifying assumptions and conditions. A number of these conditions have already been stated. We continue the process of simplification. For the sake of simplicity we will assume the following:

(1) Both objects are particles with masses m_1 and m_2. Hence, the charges Q_1 and Q_2 are point-like.

(2) The charges Q_1 and Q_2 have the same sign.

(3) The effect produced by Earth's electric field can be ignored.

(4) The object of mass m_2 carrying charges Q_2 is fixed to Earth's surface, and the object of mass m_1 carrying charge Q_1 is positioned right above object m_2 at an altitude h (above the surface of Earth).

(5) Altitude h is small compared with Earth's radius R. This means that we can ignore all variations of the accel-

eration of free fall g within this altitude (i.e. g = 9.8 m/s^2 = const).

(6) The initial velocity of m_1 is zero ($v_{01} = 0$).

(7) Air drag is negligible.

Thus, we are ready to formulate the terms of the first and simplest problem.

Problem 1. *A particle of mass m_2 carrying a charge Q_2 is fixed on the surface of Earth. Another particle, mass m_1 and charge Q_1, is positioned right above the first one at an altitude $h \ll R$, with R the radius of Earth. Charges Q_1 and Q_2 have the same sign. Determine the velocity of m_1 at a distance h_1 from the surface of Earth if the initial velocity of m_1 was zero. Air drag and Earth's electric field are ignored.*

Solution of Problem 1. Applying the law of energy conservation to the closed system consisting of Earth and particles m_1 and m_2, in which only conservative forces (the force of gravity and the Coulomb force) operate, we obtain

$$m_1gh + \frac{Q_1Q_2}{4\pi\varepsilon_0 h} = m_1gh_1 + \frac{m_1v^2}{2} + \frac{Q_1Q_2}{4\pi\varepsilon_0 h_1}.$$

This yields the sought velocity:

$$v = \sqrt{2g(h-h_1) - \frac{2Q_1Q_2(h-h_1)}{4\pi\varepsilon_0 m_1 hh_1}}. \qquad (35.1)$$

Analysis and statement of other problems. Let us assume that $h_1 \ll h$. Then formula (35.1) assumes the form

$$v = \sqrt{2gh - \frac{2Q_1Q_2}{4\pi\varepsilon_0 m_1 h_1}}. \qquad (35.2)$$

Analyzing this formula, we can state, for example, the following problem.

Problem 2. *Remaining within the terms of Problem 1, find the magnitude of charge Q_2 at which the velocity of m_1 at altitude h_1 is zero.*

Solution of Problem 2. Formula (35.2) immediately yields the solution to Problem 2:

$$Q_2 = \frac{4\pi\varepsilon_0 g m_1 h_1 h}{Q_1}.$$

Performing the necessary calculations (assuming that $m_1 = 10^{-3}$ kg, $h_1 = 10$ cm, $h = 10$ m, and $Q_1 = 10^{-8}$ C), we obtain $Q_2 = 10^{-4}$ C. The electric field strength generated by such a charge at altitudes h_1 and h can be found from the following formulas:

$$E_2' = \frac{Q_2}{4\pi\varepsilon_0 h_1}, \quad E_2'' = \frac{Q_2}{4\pi\varepsilon_0 h}.$$

Substituting the numerical values yields $E_2' \approx 9 \times 10^6$ V/m and $E_2'' \approx 9 \times 10^4$ V/m, which are, indeed, much higher than the electric field strength of Earth, roughly 130 V/m.

We can now formulate the following problem.

Problem 3. *What will happen to object m_1 if its velocity vanishes at altitude h_1? At what altitude h_2 will object m_1 be in equilibrium and what will be the nature of the object's oscillations if it is disturbed from equilibrium?*

We can then formulate hundreds of problems by lifting or varying the conditions just stated. All these problems, however, are only particular cases of the following

Generalized problem. *An object of mass m carrying a charge Q is moving in an arbitrary electric field and an arbitrary gravitational field. Determine the nature of its motion.*

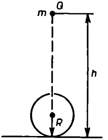

Figure 35.1

It is important to note that theoretically this generalized problem can be solved by both the dynamical method and the conservation-law method. This means that all the particular problems as well can be solved by these methods. Suppose that we have the following

Particular problem. *Electrically charged drops of mercury fall from an altitude h into a spherical metal vessel of radius R in the upper part of which there is a small opening. The mass of each drop is m and the charge on the drop is Q (Figure 35.1). What will be the number n of the last drop that can still enter the sphere?*

Solution of the particular problem. One can easily see that the given problem is a particular case of Problems 2 and 3. Each drop of mercury reaching the vessel increases

the charge of the vessel by Q. This charge is evenly distributed over the outer surface of the sphere and generates an electric field. As is known, the electric field of a uniformly charged sphere is equivalent to the field created by a point charge of the same magnitude but positioned at the center of the sphere. Thus, $Q_2 = nQ$, and Q_1 is the charge of the $(n + 1)$st drop, which is in a state of equilibrium at an altitude h above the surface of the Earth. The fact that the sum of the force of gravity mg on the $(n + 1)$st drop and the Coulomb force with which charge Q_2 acts on the charge Q of the $(n - 1)$st drop is zero implies the validity of the following equation for n:

$$\frac{nQ^2}{4\pi\varepsilon_0 (h-R)^2} = mg.$$

Here, in accordance with item (3) in the simplifying assumptions applied to the terms of Problem 1, we ignored the electric field of Earth. Hence, a meaningful formulation of the mercury-drop problem requires such values of R, h, m, and Q that satisfy the given assumption (and other assumptions as well).

36. Research Problems

Although the problems discussed above belong to the category of nonspecified and nonidealized problems, the "crux" of the problems is not clearly evident. Let us consider a problem (we will call it Example 36.1) that not only is nonspecified and nonidealized but contains the idea of the problem. In analyzing this problem we will see that it is a particular case of the nonspecified problem considered in Example 35.1 (the motion of a charged object in an electric field and a gravitational field), but, in contrast to the generalized problem in Example 35.1, we will formulate a concrete problem.

An approach to the statement of the problem. It is well-known that a gas belonging to the atmosphere of a planet escapes the planet's gravitational pull. Over a certain period of time the planet may lose its entire atmosphere (as was the case with the Moon, for example). This phenomenon can be explained by the fact that the

molecules in the atmosphere are distributed in velocity according to the Maxwell law (precisely, the Maxwell-Boltzmann law). The result is that in the atmosphere there are always molecules whose velocity exceeds the escape velocity for the planet. Such molecules may overcome the gravitational pull of the planet and leave the planet's atmosphere for ever. In place of the molecules that have escaped from the gravitational pull there come other molecules whose velocity exceeds the escape velocity, and they too will leave the atmosphere. If the gravitational pull is not very great, the process will continue until the planet loses its entire atmosphere.

How easily molecules acquire the escape velocity. The line of reasoning went like this: the escape velocity, escape from gravitational pull, the force of gravity. But other fields, too, may be used to overcome the gravitational field, say the electric field. It is well-known that Earth has an electric field whose strength at Earth's surface is about 130 V/m. Can this field be used to impart to a charged object the escape velocity?

It can be calculated that if an electron passes a potential difference of only one volt, its velocity will become approximately 600 km/s, which considerably exceeds the escape velocity for Earth (roughly 11.2 km/s). A proton after passing a potential difference of 100 V acquires a velocity of 14 km/s, which also exceeds the escape velocity. Thus, even protons accelerated in this manner can easily leave Earth.

Statement of problem. Study the possibility of using an electric field for launching a spaceship.

Approach to the statement of a research problem. For the planet we take a hypothetical planet with Earth's parameters. We assume that Earth is a ball of radius $R \approx 6400$ km and mass $M = 6 \times 10^{24}$ kg. Let us make some estimates, that is, numerical calculations of the order of magnitude of certain quantities. We start with the electric charge Q and potential φ of Earth:

$$Q = E 4\pi\varepsilon_0 R^2, \quad Q \approx 5.9 \times 10^5 \text{ C};$$

$$\varphi = \frac{Q}{4\pi\varepsilon_0 R}, \quad \varphi \approx 8.3 \times 10^8 \text{ V}.$$

We know (this can be verified by calculations) that a proton, with a charge $Q_p = 1.6 \times 10^{-19}$ C and a mass $m_p \approx 1.67 \times 10^{-27}$ kg, moving in the electric field generated by Earth's charge, may easily acquire a velocity exceeding the escape velocity for Earth and fly off into outer space. We pose the following question: what must the maximal mass m be of an object carrying an electric charge equal to that of the proton and moving in the electric field of Earth so that the object may escape Earth's gravitational pull and fly off into outer space? Let us assume that the object, a particle, is launched from Earth's surface with an initial velocity $v_0 = 0$. Then, according to the law of energy conservation,

$$-G\frac{mM}{R}+\frac{QQ_p}{4\pi\varepsilon_0 R}=0, \tag{36.1}$$

with $G = 6.67 \times 10^{-11}$ N·m²/kg² the universal gravitational constant. This yields the following formula for the maximal mass of the object:

$$m=\frac{QQ_p}{4\pi\varepsilon_0 GM}, \quad m\approx 2.1\times 10^{-18} \text{ kg},$$

which is approximately the mass of a dust particle. Thus, a dust particle with a mass 2.1×10^{-18} kg carrying a charge $Q_p = 1.6 \times 10^{-19}$ C may escape from Earth's gravitational pull. But a dust particle may carry a charge much greater than the proton charge Q_p. Two questions emerge in this connection: what can the maximal charge Q_{max} carried by a dust particle (an object) be and how can such a charge be imparted to the object (the dust particle)? The answer to both questions may be obtained if we imagine the object (the dust particle) to be a small metal ball of radius r that acquires its charge from Earth, with which it is in direct contact. The electric charge flows to the ball from Earth until the potentials of both object and Earth become equal, and then the flow stops. Since the capacitance of the ball is $C = 4\pi\varepsilon_0 r$ and $C = Q_{max}/\varphi$, we obtain

$$Q_{max}=4\pi\varepsilon_0\varphi r=\frac{4\pi\varepsilon_0 rQ}{4\pi\varepsilon_0 R}=\frac{rQ}{R},$$

where φ is the potential of Earth. Let ρ be the density of the metal ball. Then its mass is

$$m = \frac{4}{3}\pi r^3 \rho. \qquad (36.2)$$

Substituting the values of the maximal charge of the ball, Q_{max}, and mass of the ball, m, into (36.1), we arrive at an equation for determining the radius r of a ball (or density ρ) that can escape from Earth's gravitational pull and fly off into outer space:

$$-G\frac{M(4/3)\pi r^3\rho}{R} + \frac{QrQ}{4\pi\varepsilon_0 R^2} = 0.$$

Whence

$$r = \frac{Q}{4\pi}\sqrt{\frac{3}{\varepsilon_0 G\rho M R}}. \qquad (36.3)$$

We assume that the density ρ of the ball is 10^3 kg/m^3 (the ball may be hollow). Carrying out the calculations in the SI system of units, we obtain $r \approx 1.8 \times 10^{-2}$ m $=$ 1.8 cm. According to formula (36.2), the mass of such a ball is about seventeen grams.

Statement of a research problem. Thus, the electric field of Earth may be used to launch a miniature spaceship of radius $r \approx 1.8$ cm and mass of about 17 g with a zero launching velocity. For real spaceships to be launched we must change (increase) the electric field of Earth. Suppose that now the radius of the spaceship is $r_1 \approx$ 180 cm $= 1.8$ m. We can now formulate the terms of the first problem.

Problem 1. *What must the electric charge Q_1 of Earth be so that a spherical spaceship of radius $r_1 = 1.8$ m and density $\rho = 10^3$ kg/m^3 can be launched with zero launch velocity if it acquires a maximal charge Q_{max} from Earth? Air drag is ignored.*

Solution. The solution is easily obtained from formula (36.3):

$$Q_1 = 4\pi r_1 \sqrt{\frac{\varepsilon_0 G\rho M R}{3}}, \quad Q_1 \approx 5.9 \times 10^7 \text{ C}.$$

The surface electric field strength of Earth will then be $E_1 \approx 13\,000$ V/m and the potential will be $\varphi_1 \approx 8.2 \times 10^{10}$ V. Formula (36.2) then yields the mass of the spaceship: $m_1 \approx 17 \times 10^3$ kg. A similar problem can also be stated for the Moon.

Let us now consider another example (we will call it Example 36.2).

Approach to the statement of a problem. As is known, in the state of weightlessness in outer space many physical phenomena proceed in a manner different from that

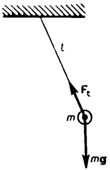

Figure 36.1

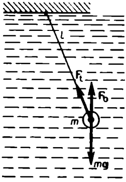

Figure 36.2

on Earth. Let us take the oscillations of a simple pendulum.

If on Earth the pendulum is placed in a vacuum, it oscillates under the force of gravity mg and the tensile stress developed by the string, $\mathbf{F}_t$ (Figure 36.1). The period of these oscillations is

$$T_0 = 2\pi \sqrt{l/g}, \tag{36.4}$$

where l is the length of the pendulum.

If the pendulum is placed in a nonviscous medium (an ideal fluid), an additional force appears, the buoyancy force $\mathbf{F}_b$ acting against the force of gravity mg (Fig-

ure 36.2). The period of oscillations in this case is

$$T_1 = 2\pi \sqrt{\frac{l}{g - F_b/m}},$$

and it becomes infinitely large at $F_b = mg$. If this happens, the tensile stress developed by the string van-

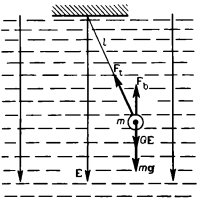

Figure 36.3

ishes (the force of gravity is compensated for by the buoyancy force).

Suppose that an electric charge Q is imparted to the pendulum, which is then placed in a homogeneous electric field **E**, as shown in Figure 36.3. The pendulum oscillates, and its period is given by the following formula:

$$T_2 = 2\pi \sqrt{\frac{l}{g - F_b/m + QE/m}}.$$

In the state of weightlessness (the pendulum is placed inside an elevator cabin that falls freely with an acceleration **g**) a force of inertia acts on the pendulum. This force $\mathbf{F}_i$ is equal in magnitude to the force of gravity $m\mathbf{g}$ but is opposite in direction (Figure 36.4). The stress $\mathbf{F}_t$ developed by the string vanishes and the pendulum does not oscillate (the force of gravity is compensated for by

the force of inertia). If a charge is imparted to the pendulum and the pendulum is placed in a homogeneous electric field, oscillations can occur even in weightlessness.

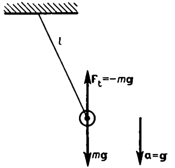

Figure 36.4

Statement of the problem. Study the possibility of using an electric field as the "medium" in which a pendulum clock can operate in weightlessness.

Statement of a research problem. The physical system consists of a simple pendulum of mass m and length l carrying a charge Q and two fields, the gravitational field and a homogeneous electric field $\mathbf{E}$. A force of inertia $\mathbf{F}_1 = -m\mathbf{g}$ acts on the pendulum. To compensate this force the electric field strength must be such that

$$mg = QE. \tag{36.5}$$

Then the period of oscillations can be calculated by formula (36.4), that is, the pendulum will oscillate with the same frequency as on Earth under ordinary conditions.

Let us imagine the bob of the pendulum to be a small ball of radius r and density ρ. The charge carried by the bob must be so small that its field can be ignored in comparison with the external electric field $\mathbf{E}$. The latter can be generated inside a parallel-plate capacitor with the distance between the plates designated by d. Suppose that the ball oscillates at a distance $d/2$ from each plate. Then the condition that the field of the charge Q on the ball be weak can be written as follows:

$$\varphi_1 = 10^{-2}\varphi, \tag{36.6}$$

where $\varphi_1 = Q/4\pi\varepsilon_0 r$ is the potential of the ball, and $\varphi = Ed/2$ the potential of the external field at the point occupied by the ball. Condition (36.6) then assumes the

form

$$\frac{Q}{4\pi\varepsilon_0 r} = 10^{-2} E \frac{d}{2},$$

or

$$Q = \frac{4\pi\varepsilon_0 r E d}{2} 10^{-2}. \qquad (36.7)$$

Substituting this into (36.5) and allowing for (36.2), we obtain

$$E = 10r \sqrt{\frac{2\rho g}{3\varepsilon_0 d}}. \qquad (36.8)$$

Calculating the value of E for $r = 10^{-3}$ m, $\rho = 10^3$ kg/m^3, and $d = 2 \times 10^{-1}$ m, we obtain $E = 6 \times 10^5$ V/m. This leads us to the following value of the potential difference across the plates: $\Delta\varphi = Ed = 1.2 \times 10^5$ V.

Apparently, it is difficult to generate such a field inside a spaceship. Therefore, we will calculate the field strength for a miniature pendulum: $r = 10^{-5}$ m, $\rho = 10^3$ kg/m^3, and $d = 2 \times 10^{-2}$ m. Then formula (36.8) yields $E \approx 2 \times 10^4$ V/m. The potential difference $\Delta\varphi \approx 400$ V. Making the radius r of the ball still smaller, we can obtain practically realizable values of the electric field strength E and the potential difference $\Delta\varphi$.

Thus, a pendulum clock operating in an electric "medium" in weightlessness must necessarily be minute. Let us formulate the terms of

Problem 1. *In a freely falling elevator cabin (on Earth) there is a parallel plate capacitor the distance between whose plates is* $d = 2 \times 10^{-2}$ m. *A simple pendulum of length* $d/2$ *is attached to the upper plate of the capacitor. The bob of the pendulum has the shape of a ball of radius* $r = 10^{-5}$ m *and density* $\rho = 10^3$ kg/m^3. *What potential difference* $\Delta\varphi$ *must be applied across the capacitor plates and what charge* Q *must be imparted to the bob if we want the pendulum to oscillate with the same frequency as it would in a motionless elevator on Earth? The electric field generated by charge* Q *carried by the bob must be weak compared to the electric field between the plates. Air drag on the pendulum is to be ignored.*

We already know the solution to this problem: it is given by formulas (36.8) and (36.7).

We can formulate other problems, say, that of considering the oscillations of a charged physical pendulum in a homogeneous electric field.

37. Arbitrary Problems

So far we have solved problems that dealt with a known topic. We usually started with the theory of the topic, then considered the basic problem, employed certain methods for its solution, suggested other methods for its solution and for solution of similar problems belonging to the same topic. Note that the solution of these problems was also carried out on the basis of a general approach developed in Part 1. In this way we moved from topic to topic, gradually acquiring experience in problem solving. This is a necessary path, but it is not at all sufficient. In life one is confronted more often with problems belonging not to one (known) topic but involving several topics, or what we call arbitrary problems.

The theory, methods, and tricks of solving physics problems already discussed are aimed at teaching the reader to solve problems whose topic is not known in advance. Now we will apply this approach to such problems.

Example 37.1. *In isothermal expansion of one mole of oxygen with a temperature* $T = 300$ K *the gas absorbed an amount of heat* $Q = 2$ kJ. *By what factor did the volume occupied by the gas increase?*

Solution. The problem is not stated quite right because the initial pressure of the gas is not given. Depending on the pressure, the gas may be considered either ideal or real. Two solutions are possible, therefore: one for a physical system consisting of an ideal gas, the other for a system consisting of a real gas.

Suppose that we are dealing with an ideal gas. Then, as a result of an isothermal process the amount of heat Q is absorbed and the amount of work

$$W = RT \ln (V_2/V_1)$$

is performed. We wish to find the ratio of two values of a macro-parameter of the system, its volume (the volume ratio V_2/V_1). This constitutes a basic problem of thermodynamics. So we employ the thermodynamic method. Applying the first law of thermodynamics, we find that

$$Q = RT \ln (V_2/V_1),$$

where we have allowed for the fact that the variation of the internal energy of an ideal gas is zero, $\Delta U = 0$. Hence,

$$\frac{V_2}{V_1} = e^{Q/RT}, \quad V_2/V_1 \approx 2\ 23.$$

Now let us assume that the physical system consists of a real gas that obeys the van der Waals equation

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT,$$

where a and b are the van der Waals constants. The internal energy of one mole of a real gas,

$$U = c_V T - a/V,$$

depends not only on temperature T but also on volume V. By the first law of thermodynamics,

$$Q = a\frac{V_2 - V_1}{V_1 V_2} + RT \ln \frac{V_2 - b}{V_1 - b}.$$

Since for oxygen $b = 0.032\ \mathrm{m^3/kmol}$, the last equation can be approximately written as

$$Q = a\frac{x-1}{V_2} + RT \ln x,$$

with $x = V_2/V_1$. To solve this transcendental equation, we must specify the final volume of the system, V_2.

Example 37.2. *A particle is moving in the positive direction along the X axis in such a manner that its velocity varies according to the law $v = a\sqrt{x}$, with a a positive constant. Assuming that at time $t = 0$ the particle was at point $x = 0$, find (a) the time dependence of the particle's velocity and acceleration, and (b) the average velocity of the particle over the time interval that it takes the particle to travel the distance from point $x = 0$ to point x.*

Solution. The physical system consists of a single object, the particle, which moves in a straight line (the one-dimensional case) along the X axis, and this motion is considered formally. An interrelationship between some of the parameters of motion (the velocity v and the position coordinate x) is fixed. We must find certain other parameters of motion as functions of time and the average velocity. This constitutes an inverse problem of kinematics.

Let us find the law of motion of the particle. Since $v_x = dx/dt$, we can write the interrelationship between velocity and the position coordinate, $v_x = a\sqrt{x}$, in the form

$$\frac{dx}{dt} = a\sqrt{x}.$$

Separating variables, integrating, and allowing for the initial conditions, we arrive at the law of motion:

$$x = a^2t^2/4. \tag{37.1}$$

This leads us to the following laws of time-variation of velocity,

$$v_x = \frac{dx}{dt} = \frac{a^2t}{2},$$

time-variation of acceleration

$$a_x = \frac{dv_x}{dt} = \frac{a^2}{2},$$

and time-variation of the average velocity,

$$\langle v_x \rangle = \frac{1}{t}\int_0^t v_x\,dt = \frac{1}{t}\int_0^t \frac{a^2t}{2}\,dt = \frac{a^2t}{4}. \tag{37.2}$$

Substituting the time of motion $t = 2\sqrt{x}/a$, which can be found from (37.1), into (37.2), we find the final expression for the average velocity:

$$\langle v_x \rangle = \frac{a}{2}\sqrt{x}.$$

Analysis of the solution shows that the problem can be made more complicated by using instead of the equation

$v_x = a\sqrt{x}$ a more general equation $v_x = f(x)$, where function $f(x)$ must satisfy several conditions.

Example 37.3. *A piston of mass M can move without friction inside a vertical cylindrical pipe whose lower end is closed. The volume under the piston is filled with a gas whose mass is considerably smaller than that of the piston. In equilibrium the distance between the lower end of the piston and the bottom of the pipe is l_0 (Figure 37.1). Find the period of the small oscillations that the piston will make if the equilibrium is disturbed, assuming that oscillations occur isothermally and the gas is ideal. The cross-sectional area of the pipe is A and the standard atmospheric pressure is p_0. Consider the limiting case when $p_0 = 0$.*

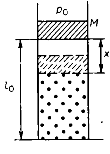

Figure 37.1

Solution. Three objects will be included in the physical system: the piston (a rigid body of mass M), the gas under the piston, and the air above the piston (an ideal gas). How will the objects of the system behave when the piston is moved downward by a distance x from the position of equilibrium (see Figure 37.1)? The state of the gas above the piston does not change (its pressure p_0 and temperature T_0 remain unchanged), while the state of the gas under the piston changes (the temperature T_0 remains unchanged, the volume decreases by $\Delta V = xA$, and the pressure increases by Δp). Hence, the piston is under an additional force $F = \Delta pA$ directed upward.

This force pushes the piston upward. In equilibrium the force F vanishes. But since the velocity of the piston at this point is nonzero, the piston moves past the equilibrium position. The distance it travels upward from the equilibrium position is also x since there is no friction. The pressure of the gas under the piston decreases by Δp. Under the force ΔpA, now directed downward, the piston moves downward. In this way the piston oscillates about the equilibrium position.

Let us employ the dynamical method. First we find the additional force $F = \Delta pA$ acting on the piston. By the Boyle law for an isothermal process involving the gas under the piston,

$$p_1 l_0 A = (p_1 + \Delta p)(Al_0 - Ax), \tag{37.3}$$

where $p_1 = (Mg + p_0A)/A$ is the pressure of the gas under the piston in the state of equilibrium. Assuming that $x \ll l_0$, we find from (37.3) the variation of the gas pressure:

$$\Delta p = p_1 x/l_0.$$

Hence, the additional force

$$F = -\frac{Mg + p_0A}{l_0}x$$

is proportional to the displacement x of the piston from the equilibrium position and is directed toward the equilibrium position.

Under this force the piston oscillates harmonically. By Newton's second law we can find the differential equation describing these oscillations:

$$M\ddot{x} + \frac{Mg + p_0A}{l_0}x = 0.$$

By comparing this equation with the general differential equation for free undamped oscillations we can find the period of the piston's oscillations:

$$T_0 = 2\pi\sqrt{\frac{Ml_0}{Mg + p_0A}}.$$

Hence, in the limiting case when $p_0 = 0$ we have $T_0 = 2\pi\sqrt{l_0/g}$, which is the period of oscillations of a simple pendulum of length l_0.

Example 37.4. *A bullet passing through a wooden board of thickness h changes velocity from v_0 to v_1. Find the time that the bullet spent in passing through the board if the resistance force is proportional to the square of velocity.*

Solution. A particle (the bullet) changes its state of motion because of a known force. The initial conditions are known: $\mathbf{v}_0 = \{v_0, 0, 0\}$ and $\mathbf{r}_0 = \{0, 0, 0\}$ at $t = 0$.

We wish to find one parameter of motion, time t. This constitutes a basic problem of particle dynamics.

Let us employ the dynamical method. By Newton's second law,

$$m\frac{dv}{dt} = -\alpha v^2, \tag{37.4}$$

where m is the bullet's mass, and α is a proportionality factor. Note that both parameters (m and α) are unknown. Integrating Eq. (37.4) and allowing for the initial condition, we arrive at the law of time-variation of the bullet's velocity:

$$v = \frac{v_0}{1+\alpha v_0 t/m}.$$

Putting $v = v_1$ in this equation, we obtain the sought time interval:

$$t_1 = \frac{m(1-v_1/v_0)}{\alpha v_1}. \tag{37.5}$$

To determine the unknown ratio m/α we find the law of motion of the bullet by solving the inverse problem of kinematics:

$$x = \frac{m}{\alpha}\ln\left(1+\frac{\alpha v_0}{m}t\right),$$

or

$$h = \frac{m}{\alpha}\ln\left(1+\frac{\alpha v_0}{m}t\right). \tag{37.6}$$

Expressing the ratio m/α by employing Eq. (37.5),

$$\frac{m}{\alpha} = \frac{v_1 t_1}{1-v_1/v_0},$$

and substituting it into (37.6), we arrive at the final expression for the sought time interval:

$$t_1 = \frac{h(v_0-v_1)}{v_0 v_1 \ln(v_0/v_1)}.$$

Example 37.5. *Two square plates with sides* $a = 300$ mm *are fixed at a distance* $d = 2.00$ mm *from each other and form a parallel-plate capacitor. They are connected to a dc source of voltage* $\Delta\varphi = 250$ V. *The plates are positioned*

vertically and lowered into kerosene with a rate $v =$ 5.00 mm/s. *Find the current* I *that flows in the leads during immersion.*

Solution. The physical system consists of the parallel-plate capacitor connected to the dc voltage source. Prior to immersion, the charge on one plate was

$$Q = C\Delta\varphi,$$

where $C = \varepsilon_0 a^2/d$ is the capacitance of the capacitor.

When the plates are being immersed, an electric current flows in the circuit. Why? Kerosene is an insulator (the dielectric constant $\varepsilon = 2$). The appearance of kerosene between the plates changes the electric field in the capacitor. This leads to a redistribution of charges on the plates of the capacitor and current begins to flow in the leads. The build-up of charge on the plates at a constant voltage ($\Delta\varphi = \text{const}$) is caused by the growth in the capacitor's capacitance C.

Let us find C at an arbitrary moment in time. By this time the plates will be immersed to a depth $h = vt$. The capacitor may be imagined as consisting of two parallel-plate capacitors connected in-parallel, one with an insulator between the plates and the other with air ($\varepsilon \approx 1$). The capacitance of this system is

$$C = \frac{\varepsilon_0 \varepsilon v t a}{d} + \frac{\varepsilon_0 (a - vt) a}{d} = \frac{\varepsilon_0 a}{d}\,[(\varepsilon - 1) vt + a].$$

The charge Q on a capacitor plate changes in time t according to the law

$$Q = \frac{\varepsilon_0 a\, \Delta\varphi}{d}\,[(\varepsilon - 1) vt + a].$$

This gives us the following formula for the sought current:

$$I = \frac{dQ}{dt} = \frac{\varepsilon_0 a\, \Delta\varphi}{d}\,(\varepsilon - 1)\, v.$$

Substituting of numerical values yields $I = 1.7 \times 10^{-9}$ A.

Example 37.6. *A spool of thread lies on a horizontal plane. With what acceleration* **a** *will the axis of the spool move if the free end of the thread is pulled with a force* **F** *(Figure 37.2). How should the thread be pulled so that the spool*

will move in the direction of the tense thread? The spool moves along the surface of the horizontal plane without slippage. Find the force of friction between spool and horizontal plane.

Solution. The physical system consists of a single object, the spool, which we consider a rigid body. The

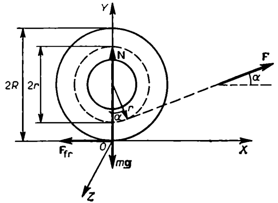

Figure 37.2

forces acting on the spool are known (they can be determined). We wish to find the acceleration of the spool. This constitutes a basic problem of rigid-body dynamics.

The following forces act on the spool: the given tensile stress **F** developed by the thread, the force of gravity mg, the force of friction $\mathbf{F}_{fr}$, and the force **N** exerted by the support on the spool. We link the inertial reference frame with Earth and direct the coordinate axes as shown in Figure 37.2. By the theorem on the motion of the center of mass,

$$ma = F\cos\alpha - F_{fr}, \quad 0 = N + F\sin\alpha - mg.$$

From the equation of motion of a rigid body about an axis passing through the center of mass we find that

$$J\beta = F_{fr}R - Fr,$$

where $\beta = a/R$ is the angular acceleration of the spool, and J the spool's moment of inertia about the rotation axis. Solving the obtained system of equations, we find the sought acceleration,

$$a = \frac{RF(R\cos\alpha - r)}{J + mR^2}, \tag{37.7}$$

and the force of friction,

$$F_{fr} = F\cos\alpha - ma.$$

From Eq. (37.7) it follows that condition $a > 0$ is met if $\cos\alpha > r/R$. To solve the problem numerically we must know the mass m and moment of inertia J of the spool.

Example 37.7. *A planoconvex glass lens touches a glass plate with its convex surface (Figure 37.3). The radius of*

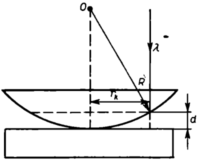

Figure 37.3

curvature of the convex surface of the lens is R, and the wavelength of the light falling on the lens is λ. Find the width Δr of a Newton ring as a function of the ring's radius in the region where $\Delta r \ll r$.

Solution. The following objects may be included in the physical system: the glass plate, the lens, and the thin air wedge between the lens and the plate (see Figure 37.3). As a result of reflection of waves from the upper and lower surfaces of the air wedge an interference pattern is formed and Newton rings appear. We wish to find the width of a ring Δr or, which is the same, find the distance between

the centers of the neighboring dark and bright rings. This constitutes a basic problem in the theory of interference of waves.

First we must find the optical path difference and then employ the maxima and minima conditions. The optical path difference of the rays reflected from the upper and lower surfaces of the air wedge is

$$\delta = 2d + \lambda/2, \tag{37.8}$$

where d is the thickness of the air wedge at the point where the rays are reflected. From geometrical considerations it follows that

$$d \times 2R = r_k^2 \tag{37.9}$$

In this equation r_k is the radius of the kth dark or bright ring. Substituting the value of d from Eq. (37.9) into Eq. (37.8) and employing the maxima condition (27.1) and the minima condition (27.2), we find the radii of the kth dark ring,

$$r_{k\mathrm{d}} = \sqrt{k\lambda R},$$

and the kth bright ring,

$$r_{k\mathrm{b}} = \sqrt{\left(k\lambda - \frac{\lambda}{2}\right) R}\,.$$

These expressions yield

$$r_{k\mathrm{d}}^2 - r_{k\mathrm{b}}^2 = (r_{k\mathrm{d}} - r_{k\mathrm{b}})(r_{k\mathrm{d}} + r_{k\mathrm{b}}) = R\lambda/2.$$

Putting $r_{k\mathrm{d}} + r_{k\mathrm{b}} \approx 2r$, we find the width of a ring:

$$\Delta r = r_{k\mathrm{d}} - r_{k\mathrm{b}} \approx \frac{R\lambda}{4r}\,.$$

Example 37.8. *An ebonite ball of radius $R = 50$ mm is charged electrically by friction with a uniformly distributed surface charge of density $\sigma = 10.0\ \mu\mathrm{C/m^2}$. The ball is rotated about its axis with a velocity $\nu = 600$ rotations per minute. Find the magnetic induction B generated at the center of the ball.*

Solution. The surface charges move in circles as the ball rotates. This generates circular currents, and around each current there is a magnetic field. We must find the

total magnetic induction of these fields at the center of the ball. This constitutes a basic problem in the theory of magnetic fields.

Let us employ the superposition principle and DI method (see Section 6). In view of the symmetry of the problem, we select a spherical system of coordinates and place its origin at the center of the ball. With planes

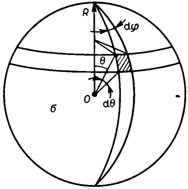

Figure 37.4

perpendicular to the rotation axis we partition the ball's surface into spherical layers so narrow that the magnetic field of the current generated by the motion of the charge carried by this layer can be calculated using the Biot-Savart law (23.3). Let us take an infinitely small surface element belonging to such a layer (in Figure 37.4 this element is hatched). The area of this surface element is $dA = R^2 \sin\theta \, d\theta \, d\varphi$, and the electric charge carried by it is $dQ = \sigma \, dA = \sigma R^2 \sin\theta \, d\theta \, d\varphi$. Since $d\varphi = \omega \, dt$, the circular current generated by the motion of the charges on this layer is

$$I = \frac{dQ}{dt} = \sigma\omega R^2 \sin\theta \, d\theta. \tag{37.10}$$

As is known (this result can easily be obtained), a circular current I of radius r generates a magnetic field whose induction B at a point lying on the axis of this current

at a distance d from the current's plane can be calculated by the formula

$$B = \mu_0\mu \frac{Ir^2}{2(r^2+d^2)^{3/2}}.$$

Thus, the circular current (37.10) generates a magnetic field whose induction at the ball's center is

$$dB = \mu_0\mu \frac{\sigma\omega R \sin^3\theta \, d\theta}{2}. \qquad (37.11)$$

Integrating (37.11) with respect to θ from 0 to π, we find that

$$B = \int_0^{\pi} \mu_0\mu \frac{\sigma\omega R \sin^3\theta \, d\theta}{2} = \frac{2\mu_0\mu\sigma\omega R}{3}.$$

Allowing for the fact that $\omega = 2\pi\nu$, we arrive at the final expression for the sought induction

$$B = 4\pi\mu_0\mu\sigma\nu R/3.$$

Substituting the numerical values, we get $B \approx 2.6 \times 10^{-11}$ T.

CONCLUSION

It is time to summarize. We have used many examples to illustrate the simple fact that the general approach to the solution of any problem in the course of college physics amounts basically to the ability to analyze an arbitrary physical phenomenon or a collection of phenomena. The fundamental concept of a physical phenomenon is linked to the majority of generalized concepts of physics: a physical system, a physical quantity, a physical law and its main aspects (the physical meaning, conditions in which the law is valid, the method of application), an interaction, the state of a physical system, a basic problem, idealized physical objects and processes, a physical model, etc. It is important not only to know all these concepts but to know how to manipulate and use them as elements of the various methods. In this system the two most general methods are the most important, the method of analyzing the physical content of a problem and the problem statement method. The first makes it possible to solve any formulated problem in the college physics course, while the second helps not only to find an approach to the solution of a nonspecified problem and to formulate and solve the "first" problem but, also, via simplification and complication, to formulate and solve dozens of problems of varying degrees of difficulty, that is, to consider a "cluster" of problems.

Above we noted that formulated and nonspecified problems exhaust the entire spectrum of problems. So it would seem that we have the answer to the question as to how to learn to solve physics problems. Basically one must master these two methods. How simple! But to do so requires a lot of hard work. In study as in science there is no easy way to achieve something.